Display
D

UNIVERSITY
OF MICHIGAN
NOV 5 1952
MATHEMATICS
LIBRARY

Mathematical Reviews

Edited by

W. Feller **E. Hille** **H. Whitney**

J. V. Wehausen, *Executive Editor*

Vol. 13, No. 9 **October, 1952** **pp. 809–896**

TABLE OF CONTENTS

History 809
Foundations 811
Algebra 813
Abstract algebra 814
Theory of groups 817
Number theory 821
Analysis 826
Calculus 827
Theory of sets, theory of functions of real variables 828
Theory of functions of complex variables . 831
Theory of series 835
Fourier series and generalizations, integral transforms 838
Polynomials, polynomial approximations . 841
Special functions 842
Differential equations 843
Functional analysis, ergodic theory . . . 848
Theory of probability 850
Mathematical statistics 853
Mathematical economics 855
Topology 856
Geometry 861
Convex domains, extremal problems, integral geometry 863
Algebraic geometry 865
Differential geometry 867
Numerical and graphical methods . . . 871
Astronomy 875
Mechanics 876
Hydrodynamics, aerodynamics, acoustics . 877
Elasticity, plasticity 883
Mathematical physics 890
Optics, electromagnetic theory 890
Quantum mechanics 891
Thermodynamics, statistical mechanics . . 894

AUTHOR INDEX

van Aardenne-Ehrenfest, T.-de Bruijn, N. G. 857
Abelé, J. 876
Abramov, A. A. 869, 870
Adams, E. N., II 895
Agmon, S. 837
Agostinelli, C. 843
Aĭdezer, N. I. 826
Albert, A. A. 816
Amante, S. 861
Amir (Jakimovski), A. 835
Ammeter, H. 853
Anderson, R. D.-Klee, V. L., Jr. 863
Andreotti, A. 866
'Arafat, W. See Winter, H. J. J.
Aronszajn, N. 830
Auger, L. 810
Bagchi, H. D.-Chatterjee, P. C. 842
Baiada, E. 831
Balk, M. B. 863
Banachiewicz, T. 872
Bar-Hillel, Y. 812
Barnes, E. S. 825
Batchelor, G. K. 879
Bays, S. 817
Benedicty, M. 866
Berman, J. H. See Nelson, H. C.
Bernkeln, S. N. 842
Besicovitch, A. S. 831
Bhatnagar, P. L. See Kushwaha, R. S.
Birger, I. A. 888
Birkhoff, G. 877
Bizley, M. T. L. 851
Blambert, M. 837
Blau, J. H. 830
Bloh, V. I. 885
Blohincev, D. I. 894
Boas, R. P., Jr. 838
Bocheński, I. M. 811
Bochner, S. 823, 834
Bogolyubov, N. N. 810
Boldyreff, A. W. 822
Bondi, C. M.-Bondi, H. 875
Bondi, H. See Bondi, C. M.
Bouligand, G. 869, 868
Brauer, A. 813
Bridgman, P. W. 812
Brock, P.-Murray, F. J. 873
de Broglie, L. 894
Brooks, J. E.-Domb, C. 896
Brownell, F. H. 847
de Bruijn, N. G. 830
de Bruijn, N. G. See van Aardenne-Ehrenfest, T.
de Bruijn, N. G.-Erdős, P. 836
Bruins, E. M. 861
Brusotti, L. 866
Buchholz, H. 890
Bückner, H. 875
Bulgakov, B. V. 843
Bultot, F. 883
Bulygin, V. Ya. 888
Bureau, F. 847
Byrd, P. F. 878
Cacciopoli, R. 830
Caligo, D. 886
Campbell, J. G. 844
Cansado, E. 853
Cartwright, M. L. 810
Castoldi, L. 841, 869
de Castro Brzezicki, A. 844
Cattaneo, C. 868
Cavallaro, V. G. 861
Cavé, R. 853
Čelidze, V. G. 836
Cesari, L. 831
Chakaya, D. G. 809
Chakravarty, N. K. 842
Chandra, D. 843
Chandrasekhar, S.-Elbert, D. 871
Chandrasekhar, S.-Elbert, D.-Franklin, A. 871
Chatterjee, P. C. See Bagchi, H. D.
Chehata, C. G. 817
Cheo, L. 823
Chern, Shiing-shen. 864
Chern, Shiing-Shen.-Chevalley, C. 810
Cherubino, S. 815
Chevalley, C. See Chern, Shiing-Shen.
Choquet, G. 829
Cicala, P. 884
Cinquini, S. See Cinquini Cibrario, M.
Cinquini Cibrario, M.-Cinquini, S. 845
Clark, R. A.-Reissner, E. 885
Clendenin, W. W. See Meyerott, R. E.
Cohen, E. 823
Cohn, R. M. 815, 816
Collinder, B. 851
Conway, H. D. 884
Coolidge, J. L. 809
Corrsin, S. 879
Costa de Beauregard, O. 892
Couchet, G. 880
Cowling, V. F. 832
Cox, D. R. 854
Cramer, H. 853
Crespo, R. 810
Čudakov, N. G.-Rodoskiĭ, K. A. 824
Čunihin, S. A. 818
Curry, H. B. 811
Curtiss, C. F.-Hirschfelder, J. O. 873
Dadda, L. 875
Dahr, K. 891
Dantoni, G. 865
van Dantzig, D. 850
Davenport, H.-Erdős, P. 825
Davis, P.-Pollak, H. 842
De Cicco, J. See Kasner, E.
Dedò, M. 862
Deemer, W. L.-Olkin, I. 855
Dénes, P. 822
Derwidué, L. 866
Descombes, R.-Poitou, G. 825
Deverall, L. I.-Thorne, C. J. 842
Dieudonné, J. 820, 849
Dirac, G. A. 856, 857

(*Continued on cover 3*)

ONE-SIDE EDITION OF MATHEMATICAL REVIEWS

An edition of MATHEMATICAL REVIEWS printed on only one side of the paper is available to persons interested in making card files of reviews or wishing to add remarks to reviews in the future. This special edition may be obtained for an additional payment of $1.00. A regular current subscription can be changed to a one-side subscription by paying the additional $1.00. This edition is folded but not stitched.

MATHEMATICAL REVIEWS

Published monthly, except August, by

THE AMERICAN MATHEMATICAL SOCIETY, Prince and Lemon Streets, Lancaster, Pennsylvania

Sponsored by

THE AMERICAN MATHEMATICAL SOCIETY
THE MATHEMATICAL ASSOCIATION OF AMERICA
THE INSTITUTE OF MATHEMATICAL STATISTICS
THE EDINBURGH MATHEMATICAL SOCIETY
SOCIÉTÉ MATHÉMATIQUE DE FRANCE
MATEMATISK FORENING I KØBENHAVN
HET WISKUNDIG GENOOTSCHAP TE AMSTERDAM
THE LONDON MATHEMATICAL SOCIETY
POLISH MATHEMATICAL SOCIETY
UNIÓN MATEMÁTICA ARGENTINA
INDIAN MATHEMATICAL SOCIETY
UNIONE MATEMATICA ITALIANA

Editorial Office

MATHEMATICAL REVIEWS, 80 Waterman St., Providence 6, R. I.

Subscriptions: Price $20 per year ($10 per year to members of sponsoring societies). Checks should be made payable to MATHEMATICAL REVIEWS. Subscriptions may be addressed to MATHEMATICAL REVIEWS, Lancaster, Pennsylvania, but should preferably be addressed to the American Mathematical Society, 80 Waterman St., Providence 6, R. I.

This publication was made possible in part by funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. Its preparation is also supported currently under a contract with the Office of Scientific Research, Headquarters, Air Research and Development Command, U. S. Air Force. These organizations are not, however, the authors, owners, publishers, or proprietors of this publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Entered as second-class matter February 3, 1940 at the post office at Lancaster, Pennsylvania, under the act of March 3, 1879. Accepted for mailing at special rate of postage provided for in paragraph (d-2), section 34.40, P. L. and R. of 1948, authorized November 9, 1940.

Mathematical Reviews

Vol. 13, No. 9 OCTOBER, 1952 Pages 809-896

HISTORY

*Neugebauer, O. **The exact sciences in antiquity.** Einar Munksgaard, Copenhagen; Oxford University Press, London, 1951; Princeton University Press, Princeton, N. J., 1952. xvi+191 pp. (14 plates). $5.00.

This book originated from six lectures delivered by the author at Cornell University in 1949 and consequently does not aim at an exhaustive discussion of the vast subject. The main emphasis is laid on mathematics and astronomy in Babylonia and Egypt in their relationship to Hellenistic science. Ch. I deals with the writing of numbers in antiquity and with some elementary arithmetic, ch. II with Babylonian mathematics. In ch. III the technical treatment is interrupted by a discussion of the way in which the source material for the investigation of ancient science is brought to light and made available to scholars, and of the dangers it is exposed to. Ch. IV resumes the thread of the exposition with a concise sketch of Egyptian mathematics and astronomy. In ch. V the author enters more closely into a discussion of Babylonian astronomy. In ch. VI the puzzling question of the origin and transmission of Hellenistic science is tackled. To each chapter a bibliography and a most valuable collection of notes and references is annexed.

E. J. Dijksterhuis (Oisterwijk).

Nobile, Vittorio. Il conflitto fra copernicisti e aristotelici nella sua essenza e nel pensiero di Galileo. Rilievi e precisazioni. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **10**, 337–343 (1951).

Nobile, Vittorio. Il conflitto fra copernicisti e aristotelici nella sua essenza e nel pensiero di Galileo. Rilievi e precisazioni. III. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **11**, 311–319 (1951).

For part I see the same Rend. (8) **9**, 299–306 (1950); these Rev. **12**, 577.

Winter, H. J. J., and 'Arafat, W. A discourse on the concave spherical mirror by Ibn al-Haitham. J. Roy. Asiatic Soc. Bengal. Sci. **16**, 1–16 (1950).

This is a translation of a medieval Arabic work which discusses the theoretical and practical problems involved in the construction of burning mirrors whose reflecting surface is the interior of a spherical zone. *E. S. Kennedy.*

Winter, H. J. J., and 'Arafat, W. The algebra of 'Umar Khayyām. J. Roy. Asiatic Soc. Bengal. Sci. **16**, 27–77 (1950).

This is an additional translation of the algebraic work of the 9th century Iranian astronomer and mathematician, Khayyam. The work is devoted mainly to a classification of cubic equations having real coefficients and to finding their positive roots by means of intersecting conics. The translation under review does not differ essentially from that of D. S. Kasir [The algebra of Omar Khayyam, Columbia Univ., 1931], and both English versions have made use of the edition of Woepcke [L'algèbre d'Omar Alkhayyâmî, . . . , Duprat, Paris, 1851]. It is particularly regrettable that duplication of effort should occur in a field in which the vast bulk of the manuscript material has remained untouched. *E. S. Kennedy* (Beirut).

Chakaya, D. G. Some Georgian mathematicians of the XVII-XVIIIth century. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **16**, 277–288 (1948). (Georgian. Russian summary)

This paper is devoted to two mathematical manuscripts of the 17–18th centuries in the Georgian language. One of them is a compilation of the fundamentals of arithmetic written by Dimitrī Cicišvili. Of the Georgian mathematical manuscripts which have come down to us the work of Cicišvili is the first document of the knowledge in Georgia of European mathematics. The second manuscript is an original textbook on arithmetic and geometry prepared by Georgī Tarhnišvili. The author of the present paper establishes that in Georgia of the 17–18th centuries both the preparation of original textbooks and compilations introducing Russian and European mathematical literature were undertaken. *Author's summary.*

Švecov, K. I. Slavonic numeration. Mat. v Skole **1952**, no. 2, 8–12 (1952). (Russian)

Coolidge, J. L. The unsatisfactory story of curvature. Amer. Math. Monthly **59**, 375–379 (1952).

Sergescu, Pierre. Sur les relations mathématiques franco-roumaines. Bull. Sci. Roumain **1**, 5–17 (1952).

Bouligand, G. Epistémologie dualiste et analyse historique de quelques théories. Thalès **6** (1949–1950), 1–23 (1951).

*Doig, Peter. **A concise history of astronomy.** Philosophical Library, New York, N. Y., 1951. xi+320 pp. $4.75.

No general history of astronomy has appeared in English in about forty years; the last such probably was W. W. Bryant's "A history of astronomy" [Methuen, London, 1907]. No wonder that Sir Harold Spencer Jones, the Astronomer Royal, in a preface welcomes this book by the editor of the Journal of the British Astronomical Association. It contains four chapters on early astronomy, two on the time from Copernicus to Newton, one on Newton, two on the eighteenth century, one on William Herschel, three on the nineteenth and four on the twentieth century. The story goes back far enough to discuss the orientation of Stonehenge and is sufficiently up to date to include Baade's theory of two types of stellar populations (1944) and Babcock's observations on the magnetic fields of rapidly rotating stars (1947). The book is fairly short and therefore considerably less comprehensive than the new book by A. Pannekoek "The growth of our world picture" [Wereld, Amsterdam, 1951 (in Dutch)]. But the concise character of the history

here makes for simplicity and clarity in exposition, and astronomers as well as non-astronomers will appreciate this book as a very readable orientation in the achievements of past and present. The mathematical side of astronomy is somewhat sacrificed for material easier grasped by the reader. This may occasionally have led to oversimplification, as in the exposition of Chaldean astronomy and of Ptolemy's theory, or in the omission of Hill's lunar theory. There are no pictures or figures. *D. J. Struik* (Cambridge, Mass.).

Cartwright, M. L. Non-linear vibrations: a chapter in mathematical history. Math. Gaz. **36**, 81–88 (1952).

Presidential address to the Mathematical Association, January 3, 1952.

Labrador, Juan Francisco. Maria Cayetana Agnesi. Gaceta Mat. (1) **3**, 175–178 (1 plate) (1951). (Spanish)

Graves, L. M. Obituary: Gilbert Ames Bliss, 1876-1951. Bull. Amer. Math. Soc. **58**, 251–264 (1952).

A list of Bliss's published mathematical papers and books and of dissertations which he supervised is included.

Perron, Oskar. Obituary: Harald Bohr. Jber. Deutsch. Math. Verein. **55**, Abt. 1, 77–88 (1952).

Perron, Oskar. Obituary: Constantin Carathéodory. Jber. Deutsch. Math. Verein. **55**, Abt. 1, 39–51 (1952).

Chern, Shiing-Shen, and Chevalley, Claude. Obituary: Elie Cartan and his mathematical work. Bull. Amer. Math. Soc. **58**, 217–250 (1952).

A complete list of Cartan's mathematical papers since 1939, as well as a partial list of all his papers, is included. E. Cartan was born April 9, 1869 and died May 6, 1951.

Garnier, René. Notice nécrologique sur M. Guido Castelnuovo. C. R. Acad. Sci. Paris **234**, 2241–2244 (1952).

Auger, L. Polémique entre Descartes et Gilles Personne de Roberval. Thalès **6** (1949–1950), 59–67 (1951).

Sarmento de Beires, Rodrigo. Evocation of the life and work of Professor Gomes Teixeira. Anais Fac. Ci. Porto **35**, 173–192 (1 plate) (1951). (Portuguese)

***Pamyati S. V. Kovalevskoĭ. Sbornik statei. [In remembrance of S. V. Kovalevskaya. A collection of essays.]** Izdat. Akad. Nauk SSSR, Moscow, 1951. 155 pp. (1 plate). 8.6 rubles.

This volume contains the following essays. The life and scientific activity of S. V. Kovalevskaya (pp. 7–66) by P. Ya. Polubarinova-Kočina. The social and literary activity of S. V. Kovalevskaya (pp. 67–97) by M. V. Nečkina. S. V. Kovalevskaya and P. L. Čebyšev (pp. 98–119) by V. E. Prudnikov. A. N. Stranolyubskiĭ—pedagogue and mathematician (pp. 120–132) by V. E. Prudnikov. On the first presentation of the drama, "The struggle for happiness", by S. Kovalevskaya and A. C. Leffler (pp. 133–143) by T. L. Ščepkina-Kupernik. Recollections of Sof'ya Vasil'evna Kovalevskaya (pp. 144–154) by S. Vl. Kovalevskaya.

Bogolyubov, N. N. Mikolaĭ Mitrofanovič Krylov. (For his seventieth birthday.) Ukrain. Mat. Žurnal **2**, no. 3, 3–6 (1 plate) (1950). (Russian)

Luzin, N. N. Works on the theory of functions of a complex variable. Uspehi Matem. Nauk (N.S.) **7**, no. 2 (48), 3–6 (1952). (Russian)

"Extract from a letter from N. N. Luzin to A. I. Markuševič sent in 1947 as material for an essay in the collection 'Matematika za XXX let' " [cf. these Rev. **11**, 148].

Fëdorov, V. S. The works of N. N. Luzin on the theory of functions of a complex variable. Uspehi Matem. Nauk (N.S.) **7**, no. 2(48), 7–16 (1952). (Russian)

Gol'cman, V. K., and Kuznecov, P. I. The works of N. N. Luzin on differential equations and on numerical methods. Uspehi Matem. Nauk (N.S.) **7**, no. 2(48), 17–30 (1952). (Russian)

✓***The scientific papers of James Clerk Maxwell.** Edited by W. D. Niven. Dover Publications, Inc., New York, 1952. xxxii+607+viii+806 pp. $10.00.

The two volumes of the original edition [Cambridge University Press, 1890] are here reproduced by photo-offset and bound together in one volume.

Lyapunov, A. A. Pëtr Sergeevič Novikov (for his 50th birthday). Uspehi Matem. Nauk (N.S.) **7**, no. 2(48), 193–196 (1 plate) (1952). (Russian)

Rybkin, G. F. Materialistic features of the Weltanschauung of M. V. Ostrogradskiĭ and his teacher T. F. Osipovskiĭ. Uspehi Matem. Nauk (N.S.) **7**, no. 2(48), 123–144 (1952). (Russian)

Masotti, Arnaldo. In memoria di Gabrio Piola nel centenario della morte. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **14** (83), 695–723 (2 plates) (1950).

Del matematico milanese Gabrio Piola (1794–1850), che fu presidente dell'Istituto Lombardo, si rammenta brevemente la vita, e si passano in rassegna le opere.

Author's summary.

✓***Oeuvres de Henri Poincaré.** Publiées sous les auspices de l'Académie des Sciences par la Section de Géométrie. Tome VII. Publié avec la collaboration de Jacques Lévy. Gauthier-Villars, Paris, 1952. viii+635 pp.

This volume includes Poincaré's various publications on analytical and celestial mechanics, grouped as follows: equilibrium of rotating fluid masses; principles of mechanics; problem of three bodies; series occurring in celestial mechanics. As in earlier volumes [cf. these Rev. **13**, 421], the relevant sections of Poincaré's own analysis of his work is included [Acta Math. **38**, 3–135 (1921), pp. 104–114]. Notes by J. Lévy sketch the relation to work of other authors and more recent developments.

Crespo, Ramón. Ernesto Schröder. Gaceta Mat. (1) **3**, 211–214 (1 plate) (1951). (Spanish)

FOUNDATIONS

***Bocheński, I. M. Précis de logique mathématique.** F. G. Kroonder, Bussum, Netherlands, 1948. 90 pp.

This précis is intended both as an elementary textbook or companion on which a course of lectures for beginners may be based and as a background for the application of symbolic logic to non-mathematical subjects. In both respects, it has a certain affinity with Carnap's well-known Abriss der Logistik [Springer, Berlin, 1929], consciously so. Thus the main sections of the book deal with the calculi of propositions, of predicates, of classes, and of relations. However, while Carnap's book is based almost exclusively on the logic of Principia Mathematica, the present work aims at giving even the beginner an idea of more recent developments. Prominence is given to the symbolism of Lukasiewicz, and there are sections on Gentzen's calculus of deduction and on the formalisation of syllogistic logic and brief references to many-valued and combinatorial logics. Considerable emphasis is placed on precise syntactic and semantic definitions. A beginner might find it difficult to appreciate some of these without the guidance of an instructor. The reviewer has the following suggestions to offer for use in subsequent editions. 1) By the introduction of only minor modifications in the present book it should be possible to exhibit clearly Bernays' complete system of axioms for the lower functional calculus (restricted calculus of predicates) as given in Hilbert and Ackermann [Grundzüge der theoretischen Logik, Springer, Berlin, 1928]. 2) The brief and apparently innocuous section 12.1 in which quantified sentences are interpreted as infinite conjunctions or disjunctions really lets the devil in through a back-door. The author is of course well aware of the problems associated with the introduction of the symbol $\cdots$ (et cetera) into a formal calculus. However, for the sake of a beginner, it may be better either to omit this section or to discuss the difficulty in more detail. 3) The sentence following 20.11 is out of place. This is clearly a printer's error.

Summing up, this is a very useful little book. Its appearance is timely especially in view of the recent revival of interest in symbolic logic in the French-speaking world.

A. Robinson (Toronto, Ont.).

Rose, Alan. An extension of computational logic. J. Symbolic Logic **17**, 32–34 (1952).

This is a straightforward analogue of N. P. Levin's Computational logic [J. Symbolic Logic **14**, 167–172 (1949); these Rev. **11**, 151] for the three-valued propositional calculus. Propositions are (denoted by) strings of symbols $x, y, \cdots$ and of the number 3, with suitable brackets, e.g. xy, $x(y3)$. The interchangeability of propositions is a fundamental concept and a number of axioms are postulated for it. A v-sign is a proposition which is interchangeable, according to these axioms with certain specified propositions.

The interpretation of this theory in terms of the three-valued calculus of propositions is as follows. "3" stands for any proposition which takes always the lowest truth value; two propositions are interchangeable if they agree always in truth-value; v-signs are propositions which take always the designated truth-value; xy is a proposition which takes the lowest truth-value, except when both x and y take the lowest value (in which case xy takes the designated value) and if one of these takes the second value and the other the lowest value (in which case xy takes the second value). With this interpretation, the author shows that Wajsberg's axioms for the three-valued calculus of propositions are satisfied.

A. Robinson (Toronto, Ont.).

Rose, Alan. The degree of completeness of the m-valued Łukasiewicz propositional calculus. J. London Math. Soc. **27**, 92–102 (1952).

The author shows that if a formalization of the Łukasiewicz-Tarski m-valued propositional calculus is weakly complete and has for its only rules of procedure modus ponens and a substitution rule, then its degree of completeness is one greater than the number of divisors of $m-1$. This generalizes a previous result of the same author [same J. **26**, 47–49 (1951); these Rev. **12**, 662] for the case where $m-1$ is a prime, the degree of completeness being then three.

O. Frink (State College, Pa.).

Curry, Haskell B. The system *LD*. J. Symbolic Logic **17**, 35–42 (1952).

The systems LA, LM, LD were introduced by the author in his book "A Theory of formal Deducibility" [Notre Dame, Ind., 1950; these Rev. **11**, 487]. LA is the positive intuitionistical calculus, LM the minimal calculus, LD is LM with the law of excluded middle, all in Gentzen's logistic form. A * indicates the adjunction of quantifiers. The following theorems are proved. (i) If $\mathfrak{X}$, $\mathfrak{Y}$, A contain no negation sign, and if $\mathfrak{X}, \neg\mathfrak{Y} \vdash A$ holds in LD^*, then $\mathfrak{X} \vdash A$ holds in LA^*. (ii) If $\mathfrak{X} \vdash A$ holds in LD, then $\mathfrak{X}, \neg A \vdash A$ holds in LM; if $\mathfrak{X} \vdash$ holds in LD, then it holds in LM also.

A. Heyting (Amsterdam).

***Markov, A. A. The theory of algorithms.** Trudy Mat. Inst. Steklov., v. 38, pp. 176–189. Izdat. Akad. Nauk SSSR, Moscow, 1951. (Russian) 20 rubles.

The author considers that the Church-Kleene-Turing theory of computability touches the idea of "algorithm" so indirectly as to make an independent development desirable. After somewhat lengthy preliminaries (e.g. definition of an "occurrence" of one word in another) an "algorithm $\mathfrak{A}$ in normal form" is defined as follows. It is (or is determined by) a list of instructions in one of the forms

$$P \rightarrow Q \quad \text{or} \quad P \rightarrow \cdot Q,$$

where P and Q are words in some alphabet; either or both of P or Q may be the empty word (i.e. absent). Thus $ab \rightarrow ba$ and $a \rightarrow \cdot$ and $\rightarrow$ are possible instructions. The order of the instructions in the list must be specified. To apply the algorithm to a given word X, take the first instruction in the list whose left-hand side occurs in X (the empty word occurs before and after every letter), and replace the first occurrence of P by Q. If there is a dot after the arrow this ends the process; if not, apply $\mathfrak{A}$ in the same way to the resulting word; and so on. If the process terminates the final result is $\mathfrak{A}(X)$.

In support of the claim that this gives a satisfactory meaning of "algorithm" it is asserted that (1) every algorithm $\mathfrak{A}$ in the rather vague sense of ordinary mathematics, is equivalent to some normal algorithm $\mathfrak{N}$, i.e. applied to any given word X, $\mathfrak{N}$ terminates if and only if $\mathfrak{A}$ does, and then $\mathfrak{N}(X) = \mathfrak{A}(X)$; (2) various specified combinations of normal algorithms (iteration, juxtaposition of results, etc.) are equivalent to normal algorithms; (3) a "universal normal algorithm" $\mathfrak{U}$ exists such that, if the instructions specifying

$\mathfrak{A}$ are combined in an obvious way (with spacing symbols) into a single word $\mathfrak{A}^*$, $\mathfrak{U}(\mathfrak{A}^*P)=\mathfrak{A}(P)$ (if either exists, $\mathfrak{A}^*P$ denoting mere juxtaposition). No indications of the proofs are given. The word $\mathfrak{A}^*$, translated into a 2-letter alphabet, is the "signature" of the algorithm $\mathfrak{A}$. A normal algorithm is "self-applicable" if its application to its own signature terminates. There is no normal algorithm which terminates for just those words that are signatures of non-self-applicable normal algorithms (proved by the usual Cantor-type argument). It follows by familiar arguments that the range of $\mathfrak{U}$ (above) is not discernible, i.e. there is no normal algorithm $\mathfrak{A}$ that terminates for every X and is such that $\mathfrak{A}(X)$ is the empty word if and only if $\mathfrak{U}$ terminates when applied to X.

In spite of his initial references, the author makes no mention of the question that will occur to every reader: what is relation between "normal algorithms" and computability à la Church-Kleene-Turing? It is obvious that normal algorithms can be programmed on a computing machine, indeed they resemble instruction-tables for existing machines a good deal more closely than do the tables for Turing's original "paper" machine [Proc. London Math. Soc. (2) 42, 230–265 (1936)]. The strong resemblance between the general course of this theory and Turing's [op. cit.] makes it likely that there is full equivalence. If that is so the interest of this theory will lie in its provision of a new and particularly simple means of constructing unsolvable problems. [In the "theorem on a universal algorithm", foot of p. 185, for the first two occurrences of $\mathfrak{A}$ read $\mathfrak{U}$.]

M. H. A. Newman (Manchester).

Bar-Hillel, Yehoshua. Rational numbers as triplets of natural numbers. Riveon Lematematika 5, 53–54 (1952). (Hebrew. English summary)

Every rational number can be written in the form $(a-b)/c$ where a, b, c are positive integers. Starting from this fact one can define rational numbers as triplets of positive integers (a, b, c) instead of first defining integers as pairs of positive integers and then rational numbers as pairs of integers. This was first suggested by Fitch [J. Symbolic Logic 15, 17–24 (1950), where earlier references are given; these Rev. 12, 2]. The present note gives explicit definitions for equality, sum, product, etc. The author points out that these definitions are obvious once the interpretation of (a, b, c) is known. *A. Robinson* (Toronto, Ont.).

***Speiser, Andreas. Elemente der Philosophie und der Mathematik. Eine Anleitung zum inhaltlichen Denken.** Verlag Birkhäuser, Basel, 1952. 115 pp. 11.45 Swiss francs.

***Bridgman, P. W. The nature of some of our physical concepts.** Philosophical Library, New York, N. Y., 1952. 64 pp. $2.75.

This book indicates the interrelations between the instrumental and mental operations involved in the definition of physical concepts. Mental operations include mathematical manipulations (called paper-and-pencil operations), verbal operations, and idealized experiments. Mental operations allow free invention, but must be capable of eventually making connection with instrumental operations. The manner of making such a connection, and the difficulties involved, are illustrated by the paper-and-pencil operation of passing to a limit.

A field is posited to account for certain phenomena which would otherwise involve action-at-a-distance. But there is as yet no instrumental operation for demonstrating the existence of the field, or for showing that action-at-a-distance is inadmissable. (The use of a probe particle is shown to be inconclusive.) Thus any choice between field and action-at-a-distance must be purely subjective, such as commonsense or convenience. With regard to empty space, it is argued first that the concept is self-contradictory, and second that (in effect) no part of space is empty. There is as yet no instrumental evidence that light is a travelling thing, rather than delayed action-at-a-distance; while the one concept may be the more convenient, the other cannot be discarded.

The law $dE=dQ+dW$ is regarded first as a definition of dE. [No reference is made to units of measurement, or to Joule's constant. Apparently only mechanical units are assumed.] This law then acquires physical content by the assertion that dE is an exact differential in the variables of state. Energy and its conservation is considered to be more than a mere convention; nevertheless, energy as a state function is defined by a paper-and-pencil operation, and so must not be reified. [The concept "thing" is not defined here, and any distinction between "thing" and "change of thing" is merely insinuated.] If the law $dE=dQ+dW$ is applied to any subsystem obtained by paper-and-pencil operations, it follows that there must be flow vectors $\mathbf{q}$ and $\mathbf{w}$ such that, for unit time, $dQ=-\operatorname{div}\mathbf{q}$ and $dW=-\operatorname{div}\mathbf{w}$; these fluxes are observed instrumentally (more or less directly). It also follows that dE must be localized in space; this is consistent with the localization of energy in an electrostatic field [cf. above] as evidenced by magnetic effects, and suggests the possibility that energy is localized in a gravitational field with an as-yet-undetected effect analogous to magnetism. The law $dS=dQ/t$ is regarded first as a definition of dS. (The difficulties of distinguishing between heat and work are discussed in detail.) This law acquires physical content by asserting that dS is an exact differential in the variables of state. The discussion of dS parallels that of dE except for the creation of entropy and the need for generalizing the entropy concept. The potential jump involved in the Volta effect and in the thermoelectric effect is commonly accounted for by the paper-and-pencil operation of introducing a double layer; however, this double layer is not ordinarily uniquely determined. An exception occurs in the case of a thermocouple whose two elements are cut in perpendicular directions from the same crystal. A detailed analysis is given of the energy flows in a thermoelectric circuit. *C. C. Torrance* (Monterey, Calif.).

Weitzenhoffer, André M. Mathematical structures and psychological measurements. Psychometrika 16, 387–406 (1951).

The author discusses the question of why mathematics does not find such ready application in psychology as it does in the physical sciences. He concludes that psychological quantities do not satisfy certain postulates necessary for establishing significant correspondences with numbers.

H. B. Curry (State College, Pa.).

ALGEBRA

Riordan, John. A recurrence relation for three-line Latin rectangles. Amer. Math. Monthly 59, 159–162 (1952).

1) On sait qu'un rectangle latin est un ensemble de permutations des mêmes lettres, telles que toute lettre occupe des rangs différents dans chacune de ces permutations. Dans le cas particulier où le rectangle contient trois lignes et n lettres, soit L_n le nombre des rectangles distincts ainsi formés, soit K_n le nombre des rectangles dont la première ligne est dans l'ordre standard; alors, $L_n=n!K_n$. 2) L'auteur, dont on connaît les deux mémoires sur la question [même Monthly 51, 450–452 (1944); 53, 18–20 (1946); ces Rev. 6, 113; 7, 233] applique les formules de Yamamoto [Sûgaku 2, 159–162 (1949)]. Il en déduit les équations aux différences: $K_n=n^2K_{n-1}+(n)_2K_{n-2}+2(n)_3K_{n-3}+k_n$, et $k_n+nk_{n-1}=-(n-1)2^n$. A titre de vérification, l'élimination de k entre ces deux relations restitue celle de Kerawala [Bull. Calcutta Math. Soc. 33, 119–127 (1941); ces Rev. 4, 69], mais elles sont considérablement plus commodes pour le calcul numérique de K. 3) Au moyen des équations 2) l'auteur retrouve, par une voie aisée et rapide, la formule asymptotique de Yamamoto: $L_n=(n!/e)^3\sum a_s/(n)_s$, avec $(n)_s=n!/(n-s)!$, $s=0, 1, \cdots, n$; $a_0=1$, $a_1=-1$, et $(s+1)a_{s+1}+a_s+2a_{s-1}=0$. 4) Les nombres K et k satisfont à des congruences arithmétiques simples, comme: $K_{p+n}\equiv 2K_n \pmod p$, où p est un nombre premier. Des tables de résidus faciles à dresser pour les petites valeurs de p permettent le contrôle des tables de Kerawala et de Yamamoto.

A. Sade (Marseille).

Sade, Albert. Omission dans les listes de Norton pour les carrés 7·7. J. Reine Angew. Math. 189, 190–191 (1951).

Except for a brief historical introduction, this is identically a paper previously reviewed [Ann. Math. Statistics 22, 306–307 (1951); these Rev. 12, 665]. *J. Riordan.*

Robinson's constant. Amer. Math. Monthly 59, 296–297 (1952).

Summary of a number of papers dealing with the value of a constant defined by R. Robinson [same Monthly 58, 462–469 (1951); these Rev. 13, 200].

Niblett, J. D. A theorem of Nesbitt. Amer. Math. Monthly 59, 171–174 (1952).

Denoting column and row by s and r respectively, it is shown here that the n-rowed determinant $|{}^{n+1}C_{2r-s}|=2^{\frac{1}{2}n(n+1)}$. The binomial expansions of $(x+y)^n\pm(x-y)^n$ form the basis of the two proofs given. The second proof makes use of Sylvester's eliminant also. *H. Gupta* (Hoshiarpur).

Cherubino, Salvatore. Risoluzione senza determinanti dei sistemi lineari di equazioni. Boll. Un. Mat. Ital. (3) 7, 54–59 (1952).

Stein, P. Some general theorems on iterants. J. Research Nat. Bur. Standards 48, 82–83 (1952).

For $\lim_{n\to\infty} B^n=0$, with B a real or complex matrix, it is necessary and sufficient that there exist a positive definite Hermitian matrix H such that $H-B^*HB$ is positive definite, B^* the conjugate transpose. The Jordan form, modified so the elements off the diagonal are sufficiently small, is used in the proof. If H is not positive definite but $H-B^*HB$ is, $\lim_{n\to\infty} B^n\neq 0$. If the characteristic roots of modulus one of B have multiplicity no greater than two, $H_1-B^*H_1B$ is positive semidefinite for a nonzero Hermitian H_1.

W. Givens (Knoxville, Tenn.).

Stein, P. A note on bounds of multiple characteristic roots of a matrix. J. Research Nat. Bur. Standards 48, 59–60 (1952).

If A is a square matrix of order n with complex elements and there are m independent characteristic vectors with the same root λ, then λ lies in at least m of the circles $|z-a_{ii}|=\sum_{i\neq i}|a_{ii}|$. The proof depends on the construction in any m-dimensional subspace of a basis with the property that a component of maximum modulus of each vector can be selected in such a way that no two components have the same subscript. *W. Givens* (Knoxville, Tenn.).

Brauer, Alfred. Limits for the characteristic roots of a matrix. IV. Applications to stochastic matrices. Duke Math. J. 19, 75–91 (1952).

[The three earlier parts of this paper were published in the same J. 13, 387–395 (1946); 14, 21–26 (1947); 15, 871–877 (1948); these Rev. 8, 192, 559; 10, 231.] A stochastic matrix has non-negative elements and row sums all equal to one, hence has the characteristic (column) vector $(1, 1, \cdots, 1)$ with associated characteristic root (called "trivial") plus one. It is proved that: 1) if a_{ii} and a_{jj} are the smallest elements on the main diagonal of a stochastic matrix, then all its roots lie in or on the boundary of the oval $|z-a_{ii}|\cdot|z-a_{jj}|\leqq(1-a_{ii})(1-a_{jj})$; 2) if A is an arbitrary unreduced matrix (i.e., not of the form $T\left(\begin{smallmatrix}P & 0\\ 0 & Q\end{smallmatrix}\right)T^{-1}$ for T a permutation matrix) a root ω of A can lie on the boundary of the region G formed by all the ovals

$$|z-a_{\kappa\kappa}|\,|z-a_{\lambda\lambda}|\leqq\Big(\sum_{\mu\neq\kappa}|a_{\kappa\mu}|\Big)\Big(\sum_{\nu\neq\lambda}|a_{\lambda\nu}|\Big),\quad \kappa\neq\lambda,$$

only if it is a boundary point of each of them (previously known: ω is in or on the boundary of G); 3) all non-trivial roots of an unreduced stochastic matrix lie in the interior of the oval of 1) unless the ovals of 2) all coincide; 4) if $z=1$ is a root of multiplicity k of a stochastic matrix A, then a simultaneous permutation of the rows and columns will carry A into either a direct sum of k unreduced stochastic matrices (with $z=1$ necessarily a simple root of each) or into this form extended with additional columns of zeros and then with suitable rows.

Let A be an arbitrary matrix, $Ax=\omega x$ for x a column vector, and let h be an arbitrary row vector. Then $B=A-xh$ has the root $\omega'=\omega-hx$ instead of ω but B and A have otherwise the same roots. This device, with $\omega=x_1=\cdots=x_n=1$, is applied to obtain special and improved results on stochastic matrices. Lemmas relating the position of certain ovals of Cassini are used in some of the proofs.

W. Givens (Knoxville, Tenn.).

Givens, Wallace. Fields of values of a matrix. Proc. Amer. Math. Soc. 3, 206–209 (1952).

The field of values $F(A)$ of an $n\times n$ matrix $A=(a_{ik})$ with complex coefficients is the set of complex numbers x^*Ax/x^*x where x is an arbitrary vector. It is known that all the characteristic roots of A lie in its interior or on its boundary. It does not in general coincide with the minimum convex polygon $P(A)$ containing all the roots, but will do so for normal matrices. The field of values is here generalized to

the set $F_H(A)$ consisting of all numbers x^*HAx/x^*Hx where H is a positive definite hermitian matrix. If $H=TT^*$ and TAT^{-1} is normal then $F_H(A)=P(A)$. This is again only a sufficient condition. Necessary and sufficient for $F_H(A)=P(A)$ is that the elementary divisors corresponding to roots lying on the boundary of $P(A)$ are simple. The intersection of all possible $F_H(A)$ always coincides with $P(A)$.

O. Taussky-Todd (Washington, D. C.).

Salzer, H. E. An elementary note on powers of quaternions. Amer. Math. Monthly **59**, 298–300 (1952).

If $X=x_0+x_1i+x_2j+x_3k$, $\rho^2=x_1^2+x_2^2+x_3^2>0$, $z=x_0+i\rho$, and $z^n=a+ib$, then $X^n=a+\lambda(x_1i+x_2j+x_3k)$ where $\lambda=b/\rho$. Various proofs are given of this result, which can be used to find powers of quaternions from a table of powers of complex numbers.

I. Niven (Eugene, Ore.).

Moppert, C. F. Deduction of Cardano's formula by conformal mapping. Amer. Math. Monthly **59**, 310–314 (1952).

Murnaghan, Francis D. A generalisation of Hermite's law of reciprocity. Anais Acad. Brasil. Ci. **23**, 347–368 (1951).

The results of this paper are best expressed in terms of the plethysm of S-functions. Hermite's law of reciprocity that the concomitants of degree n of a binary m-ic are of the same number and type as the concomitants of degree m of a binary n-ic, is then expressed as $\{m\}\otimes\{n\}=\{n\}\otimes\{m\}$, for binary forms. The author's generalisation is that, for binary forms,

$$[\{m-1\}\otimes\{k\}]\{k-1\}=[\{m\}\otimes\{k-1\}]\{m-1\}.$$

Consider S-functions of the quantities z_1, z_2. Then

$$[(1-z_1^m t)(1-z_1^{m-1}z_2 t)\cdots(1-z_2^m t)]^{-1}=1+\sum\{m\}\otimes\{k\}t^k.$$

Multiplying by $(1-z_2^m t)$ and putting $r=z_2 t$, the left-hand side becomes a similar expansion with m replaced by $(m-1)$. The coefficient of r^k is $\{m-1\}\otimes\{k\}$, and comparison with the right-hand side gives

$$[\{m-1\}\otimes\{k\}]z_1^k=\{m\}\otimes\{k\}-\{m\}\otimes\{k-1\}z_2^m.$$

Interchanging z_1 and z_2, substracting, and dividing by (z_1-z_2), the required result obtains. Hermite's law is deduced by induction, putting k successively equal to m, $m+1$, $m+2$, $\cdots$. The dual result is

$$[\{m-1\}\otimes\{1^k\}]\{k-1\}=[\{m\}\otimes\{1^{k-1}\}]\{m-1\}-[\{m\}\otimes\{1^{k-2}\}]\{2m-1\}+\cdots+(-1)^k\{km-1\}.$$

The author deduces binary expansions of $\{m\}\otimes\{k\}$ and $\{m\}\otimes\{1^k\}$ for m, $k=2, 3, 4, 5, 6$.

Extension is then made to n-ary analysis of $\{m\}\otimes\{k\}$, $\{m\}\otimes\{1^k\}$. S-functions of $z_1, z_2, \cdots, z_n$ are denoted by $\{\lambda\}$ and S-functions of $z_2, z_3, \cdots, z_n$ by $\{\lambda\}'$. Operators D_r are defined such that if $f(S_1, S_2, \cdots)$ is any function of the power sums, then

$$f(S_1+z_1, S_2+z_1^2, S_3+z_1^3, \cdots)=f+(D_1f)z_1+(D_2f)z_1^2+\cdots.$$

A similar procedure to that used for binary forms then leads to the equations

$$D_1[\{1^m\}\otimes\{1^k\}]'=[\{1^m\}\otimes\{1^{k-1}\}]'\{1^{m-1}\}';$$
$$D_2[\{1^m\}\otimes\{1^k\}]'=[\{1^m\}\otimes\{1^{k-2}\}]'[\{1^{m-1}\}\otimes\{1^2\}]';$$
$$\cdots\qquad\cdots$$
$$D_k[\{1^m\}\otimes\{1^k\}]'=[\{1^{m-1}\}\otimes\{1^k\}]',$$

with similar results for $D_r[\{1^m\}\otimes\{k\}]$. This enables the terms in the expansions of $\{1^m\}\otimes\{1^k\}$ and $\{1^m\}\otimes\{k\}$ for which the partitions commence with k, to be determined in terms of plethysms of lower order, and consecutively the terms commencing with $k-1$, $k-2$, $\cdots$. The corresponding expansions for $\{m\}\otimes\{k\}$ and $\{m\}\otimes\{1^k\}$ can be deduced when $\{1^m\}\otimes\{1^k\}$ and $\{1^m\}\otimes\{k\}$ are known, by the use of the theorem of conjugates. Calculated examples give the complete expansions in various cases up to $m=8$, $k=5$.

D. E. Littlewood (Bangor).

***Papy, Georges. L'algèbre extérieure et la théorie des diviseurs élémentaires.** Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 187–191. Centre National de la Recherche Scientifique, Paris, 1950.

See Papy, Bull. Soc. Math. Belgique **1** (1947–1948), 5–14 (1949); these Rev. **11**, 155.

L. C. Hutchinson.

Papy, Georges. Sur l'arithmétique de l'algèbre de Grassmann. Bull. Soc. Math. Belgique **3** (1949–1950), 18–31 (1951).

After an introductory discussion of exterior algebra, using the notation of Bourbaki, the author notes that in ordinary vector space of dimensions $2n$, every $(n+k)$-vector is divisible by the kth power of any non-singular bivector, and asks whether this theorem be generalized to an A-module with A an integral domain of arbitrary character. The modifications necessary for such a generalization are discussed, but details of proof are apparently left for a future paper.

L. C. Hutchinson (Brooklyn, N. Y.).

***Lepage, Th. H. Sur certains idéaux de l'algèbre extérieure de degré $2n$, $n>1$.** Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 181–186. Centre National de la Recherche Scientifique, Paris, 1950.

This paper is an exposition, with improvements, of certain theorems of the author in exterior algebra of dimensions $2n$, particularly on the expression of exterior forms in terms of completely decomposable forms. Applications to matrices (Kronecker-Runge and Radon) and differential equations (characteristics of Monge and Ampère) are included. See: Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 288–299, 527–541 (1947); **35**, 325–345, 694–708 (1949); Bull. Soc. Math. Belgique **2** (1948–1949), 26–32 (1950); these Rev. **9**, 265; **11**, 308; **12**, 234.

L. C. Hutchinson (Brooklyn, N. Y.).

Abstract Algebra

Grau, A. A. A ternary operation related to the complete disjunction of Boolean algebra. Univ. Nac. Tucumán. Revista A. **8**, 121–126 (1951).

A quasi-Boolean algebra is a set Q of elements with binary operations $+$ and $\cdot$ satisfying (1) Q is a group of involutions under $+$ and $\cdot$ with identities 0 and 1; (2) $(a+b)\cdot c=a+(b\cdot c)$; (3) for each a there exists a' with $a+a'=1$ and $a\cdot a'=0$. Postulates for a ternary operation are given. Systems satisfying these postulates are stated to be in one-to-one correspondence with binary algebras; the background of this statement is obscured by typographical or editorial errors. The elements of a Boolean algebra form a quasi-Boolean algebra under the operations of complete disjunction and its dual.

P. M. Whitman (Silver Spring, Md.).

Szász, G. On the independence of a postulate system for the distributive lattices. Math. Ann. **124**, 291–293 (1952).

Proof of the independence of the usual postulates for distributive lattices. [This has already been proved by R. Croisot, Canadian J. Math. **3**, 24–27 (1951); these Rev. **12**, 472.] *P. M. Whitman* (Silver Spring, Md.).

Löwig, H. F. J. Bemerkung zu den Primquotienten eines distributiven Verbandes. J. Reine Angew. Math. **190**, 49–50 (1952).

In a distributive lattice, if a/b is a prime quotient, let $P(a/b)$ denote the prime ideal consisting of all x with $(b\cup x)\cap a=b$. Theorem: $P(a/b)=P(c/d)$ if and only if a/b and c/d are projective; then also

$$P(a/b)=P(a\cap c/b\cap d)=P(a\cup c/b\cup d).$$

P. M. Whitman (Silver Spring, Md.).

Kandô, Tetsuo. Strong regularity in arbitrary rings. Nagoya Math. J. **4**, 51–53 (1952).

An element a of a ring R is strongly regular if there is an element x of R such that $a^2x=a$; an ideal is strongly regular if each element is strongly regular. The author shows that there exists in an arbitrary ring a unique maximal strongly regular ideal, and discusses some of its properties. The methods are based on those of Brown and McCoy [Proc. Amer. Math. Soc. **1**, 165–171 (1950); these Rev. **11**, 638]. Several of the theorems are implicit in another paper of Brown and McCoy [Trans. Amer. Math. Soc. **69**, 302–311 (1950); these Rev. **12**, 474]. *N. H. McCoy.*

Iséki, Kiyoshi. Sur le G-radical d'un anneau topologique. C. R. Acad. Sci. Paris **234**, 1938–1939 (1952).

The author announces four theorems concerning the Brown-McCoy radical of a topological ring.

I. Kaplansky (Chicago, Ill.).

Northcott, D. G. A note on the intersection theorem for ideals. Proc. Cambridge Philos. Soc. **48**, 366–367 (1952).

Let $\mathfrak{a}$ be an ideal of a Noetherian ring with unity. The author characterizes $\bigcap\mathfrak{a}^i$ as the set of all elements x such that $x\equiv ax$, and also as the intersection of those primary components of (0) which belong to prime ideals which contain an element of the form $1-a(a\in\mathfrak{a})$. These results generalize a theorem of Krull and Chevalley according to which $\bigcap\mathfrak{a}^i=(0)$ if and only if no zero divisor is $\equiv 1 \pmod{\mathfrak{a}}$.

E. R. Kolchin (New York, N. Y.).

Szele, T. On ordered skew fields. Proc. Amer. Math. Soc. **3**, 410–413 (1952).

Artin and Schreier [Abh. Math. Sem. Hamburg Univ. **5**, 83–115 (1926)] proved that a field can be ordered if and only if it is formally real. This result was generalized by Serre [C. R. Acad. Sci. Paris **229**, 576–577 (1949); these Rev. **11**, 76] who obtained conditions under which the ordering of an ordered field K can be extended to an ordering of an extension field of K. In the present paper these results are generalized to the case of skew fields. The generalization of the Artin-Schreier theorem takes the following form. A skew field K can be ordered if and only if -1 cannot be expressed as a sum of elements of the form $a_1^2a_2^2\cdots a_k^2$ $(a_i\in K;\ i=1, 2, \cdots, k)$. *N. H. McCoy.*

Johnson, R. E. On ordered domains of integrity. Proc. Amer. Math. Soc. **3**, 414–416 (1952).

This paper generalizes a result of Szele [see the preceding review] as follows. Let K be a domain of integrity which, for convenience, is assumed to have a unit element. Let K^* be the set of nonzero elements of K. An element a of K^* is even if there exist elements $a_1, \cdots, a_n$ in K^* such that a is a product of the $2n$ elements $a_1, \cdots, a_n, a_1, \cdots, a_n$ in some order. If S is the additive semigroup generated by the even elements of K^*, then K can be ordered if and only if $S\subset K^*$.

N. H. McCoy (Northampton, Mass.).

Lang, Serge. Hilbert's Nullstellensatz in infinite-dimensional space. Proc. Amer. Math. Soc. **3**, 407–410 (1952).

It is shown that three different statements of the Hilbert theorem on zeros remain equivalent for a polynomial ring $k[X_\alpha]_{\alpha\in A}$ over a field k when A is permitted to be infinite, and that these statements are true if and only if either A is a finite set or the transcendence degree of k over its prime field exceeds the cardinality of A. *E. R. Kolchin.*

Wilker, P. Über die Zwischenkörper einfacher algebraischer Erweiterungen. Math. Ann. **124**, 289–290 (1952).

This paper gives an elementary, but essentially well-known proof of the classical theorem that there are only a finite number of fields between the field K and an extension field L if and only if L is a simple algebraic extension of K.

R. E. Johnson (Northampton, Mass.).

Springer, Tonny Albert. Sur les formes quadratiques d'indice zéro. C. R. Acad. Sci. Paris **234**, 1517–1519 (1952).

The author proves the following conjecture of E. Witt [J. Reine Angew. Math. **176**, 31–44 (1937)]: Let f be a quadratic form over an arbitrary field K with characteristic $\neq 2$. If f has no zeros in K, then it has no zeros in any algebraic extension L of K of odd degree. Using Witt's results [loc. cit.] in conjunction with this, the author obtains the companion theorem: If the forms f and g are not equivalent over K, then they are not equivalent over L.

W. H. Durfee (Washington, D. C.).

Flanders, Harley. Elementary proof of a norm theorem. Proc. Amer. Math. Soc. **3**, 196 (1952).

The author gives a purely field-theoretic proof of the following theorem of the reviewer. Let Y be the subfield of degree n of a cyclic field K of degree nm over F and a be a nonzero quantity of F. Then a^m is the K-norm over F of a quantity of K if and only if a is the Y-norm over F of a quantity of Y. *A. A. Albert* (Chicago, Ill.).

Cohn, Richard M. On extensions of difference fields and the resolvents of prime difference ideals. Proc. Amer. Math. Soc. **3**, 178–182 (1952).

Let F be a difference field of characteristic 0 for which there does not exist an integer $h>0$ such that every element of F equals its own hth transform. It is proved that if $\alpha_1, \cdots, \alpha_n$ are elements of an extension of F, each of which satisfies an algebraic difference equation over F, then there exists an element $\beta\in F\langle\alpha_1, \cdots, \alpha_n\rangle$, and an integer $t\geqq 0$, such that the tth transform of every element of $F\langle\alpha_1, \cdots, \alpha_n\rangle$ belongs to $F(\beta)$. The proof uses an earlier paper [Cohn, Trans. Amer. Math. Soc. **64**, 133–172 (1948); these Rev. **10**, 4] in which a weaker result was proved. Application is

made to resolvent systems of reflexive prime difference ideals of difference polynomials. *E. R. Kolchin.*

Cohn, Richard M. Extensions of difference fields. Amer. J. Math. **74**, 507–530 (1952).

Let F, G, H be difference fields of characteristic 0 with G, H extensions of F. The author calls G and H "compatible" if no extension of F contains subextensions G', H' isomorphic to G, H respectively; examples are known [Ritt, same J. **63**, 681–690 (1941); these Rev. **3**, 265]. Two reflexive prime difference ideals of difference polynomials over F (or their manifolds) are termed "incompatible" if adjunction of generic zeros of them yields two incompatible extensions; the general solution of a difference polynomial is then said to be "inconsistent" if two of its irreducible components are incompatible. The extension G is termed "monadic" if there do not exist two distinct isomorphisms of G into a single extension of F; examples are known [Cohn, Trans. Amer. Math. Soc. **64**, 133–172 (1948); these Rev. **10**, 4]. The author proves that if F has a pair of incompatible extensions (a monadic extension), finitely generated, then there exists an irreducible difference polynomial $P \in F\{y\}$ of order 0 such that the general solution of P is inconsistent (such that F has a monadic extension $F\langle\alpha\rangle$, where α is a zero of P). The paper contains various examples, remarks, and conjectures. *E. R. Kolchin* (New York, N. Y.).

Moore, John T. Division algebras over fields of formal power series. Proc. Amer. Math. Soc. **2**, 874–877 (1951).

Let K be a field of formal power series in m variables with coefficients in an algebraically closed field of characteristic 0. Using results of O. F. G. Schilling the author shows that any central division algebra D over K is a direct product of cyclic algebras of prime power degree. If D is cyclic of prime power degree, then D is primary and the exponent of D is equal to the degree. *R. Brauer* (Cambridge, Mass.).

Jacobson, N. A note on Lie algebras of characteristic p. Amer. J. Math. **74**, 357–359 (1952).

Let L be a finite-dimensional Lie algebra over a field of characteristic p and let A be the universal associative algebra of L as defined by Birkhoff and Witt. The author proves that if a is a linear element of A, then there exists a non-zero polynomial f over that ground field such that $f(a)$ is contained in the center of A. This simple lemma has important applications; it gives (1) an elegant proof of a theorem that every finite-dimensional Lie algebra of characteristic p has a faithful finite-dimensional representation [cf. K. Iwasawa, Jap. J. Math. **19**, no. 4, 405–426 (1948); these Rev. **11**, 317], (2) a proof of a conjecture of Chevalley that every finite-dimensional Lie algebra of characteristic p has a representation which is not completely reducible, (3) a proof that A can be imbedded in a division algebra. *K. Iwasawa.*

Albert, A. A. On nonassociative division algebras. Trans. Amer. Math. Soc. **72**, 296–309 (1952).

This paper is chiefly concerned with the consideration of finite non-associative quasi-division rings (i.e. rings in which division is possible). The following remarkable theorem is proved. Any strictly power associative finite quasi-division ring D of characteristic not 2 is a finite field. The phrase "strictly power-associative" means that the ring $D^{(+)}$, derived from D by taking as product $x \cdot y = \frac{1}{2}(xy+yx)$ (xy the product in D), remains power associative when the center of D is extended. This condition is equivalent to that of power-associativity if the center of D contains more than 5 elements (and has characteristic greater than 2). The proof depends on the fact (proved in the paper) that $D^{(+)}$ is then necessarily a simple Jordan algebra over its center and thence on a study of the possible structures of $D^{(+)}$.

The remainder of the paper is devoted to the study of non-associative algebras which are not power-associative. Two constructions are used to obtain such algebras. The first is that due to Dickson [Duke Math. J. **1**, 113–125 (1935)] and it is shown that this construction yields commutative central division algebras of any even order over any finite field of characteristic not 2. The second construction, which is new, yields commutative central division algebras of odd order n over any field of characteristic not 2 which possesses a cyclic extension of order n (and in particular over any finite field). In the characteristic 2 case non-commutative finite division algebras of any odd dimension are constructed. Finally, it is indicated that the existence of commutative finite division algebras of characteristic 2 is still open but that there certainly exist no such algebras of order 3 over the field containing 2 elements. *D. Rees.*

Mal'cev, A. I. On a representation of nonassociative rings. Uspehi Matem. Nauk (N.S.) **7**, no. 1(47), 181–185 (1952). (Russian)

Consider an associative algebra under the new operation $x \circ y = axy$, where a is fixed. Theorem 2 asserts that an arbitrary non-associative algebra can be exhibited as a o-subalgebra of a suitable associative algebra. One first observes (Theorem 1) that the free associative algebra generated by a and x_i gives rise to a free non-associative algebra under o, and then studies reduction modulo an ideal. Theorem 3: Any associative algebra of countable dimension can be embedded in an algebra with two generators. This theorem is unrelated but is proved by similar techniques. *I. Kaplansky* (Los Angeles, Calif.).

Nakayama, Tadasi. On construction and characterization of Galois algebras with given Galois groups. J. Reine Angew. Math. **189**, 100–117 (1951).

A preliminary account of the results of the author has appeared in the Nagoya Math. J. **1**, 11–17 (1950); these Rev. **12**, 476. The author extends Hasse's theory of Galois algebras [J. Reine Angew. Math. **187**, 14–43 (1949), these Rev. **11**, 576] to the case where the characteristic of the underlying field Ω divides the order of the Galois group G of the Galois algebra. It is assumed that the absolutely irreducible representations of G lie in Ω. An algebra $\mathfrak{A}$, not necessarily associative, is called a Galois algebra with the Galois group G, if $\mathfrak{A}$ possesses the finite group G as group of automorphisms and if $\mathfrak{A}$, considered as a right module with regard to the group algebra $\mathfrak{G}$ of G, is isomorphic to $\mathfrak{G}$ itself. It is shown that every such Galois algebra gives rise to a system $A_{\kappa\lambda}$ of matrices in Ω or rather to a class of such associated systems. Conversely, every system of matrices $A_{\kappa\lambda}$ determines a Galois algebra $\mathfrak{A}$, provided that each $A_{\kappa\lambda}$ is of the proper type. Necessary and sufficient conditions for associativity and commutativity of $\mathfrak{A}$ in terms of the $A_{\kappa\lambda}$ are given. It is remarked that the regular discriminant of $\mathfrak{A}$ can be expressed by the coefficients of the $A_{\kappa\lambda}$. In an appendix, a duality theorem of Tannaka's type is obtained. *R. Brauer* (Cambridge, Mass.).

Theory of Groups

Nakada, Osamu. Partially ordered Abelian semigroups. I. On the extension of the strong partial order defined on Abelian semigroups. J. Fac. Sci. Hokkaido Univ. Ser. I. 11, 181–189 (1951).

Let S be an Abelian semigroup which is a partially ordered set under the relation $\geqq$ such that $a \geqq b$ implies $ac \geqq bc$ for each $c \in S$. The partial order $\geqq$ on S is said to be strong, if $ac \geqq bc$ implies $a \geqq b$. It is shown that S can be imbedded in the sense of order-preserving algebraic isomorphism in a partially ordered group if and only if the partial order of S is strong. If S is linearly ordered, then for any element a of S either the powers $a, a^2, \cdots$ are all different or there is a natural integer n such that $a, a^2, \cdots, a^n$ are different but $a^n = a^{n+1} = \cdots$; in case the order of S is strong, the second alternative is impossible unless a is the identity. Several corollaries are derived from the following theorem which is the analogue of a result of the reviewer [Amer. J. Math. 72, 191–194 (1950); these Rev. 11, 323]: let P be a strong partial order defined on S and x, y any two elements of S incomparable in P; then there is a strong extension Q (and also a linear one L) of P such that $x \geqq y$ in Q (in L) if and only if P is normal, i.e. $a^n \geqq b^n$ for some positive integer n implies $a \geqq b$ (both in P). *L. Fuchs* (Budapest).

Chehata, C. G. An algebraically simple ordered group. Proc. London Math. Soc. (3) 2, 183–197 (1952).

Let F be an ordered field (e.g. the real field) and $G(F)$ the set of all transformations f of F with the following properties: 1) f is a monotone increasing mapping of F onto itself; 2) there is an interval $[\lambda, \mu]$ such that $f(x) = x$ for all $x \leqq \lambda$ and $x \geqq \mu$; 3) the interval $[\lambda, \mu]$ can be divided into a finite number of subintervals on each of which f is linear. $G(F)$ obviously forms a transformation group of F and an order-relation can be defined in $G(F)$ so that it becomes an ordered group. For any open or closed interval I of F, let $G(I)$ denote the subgroup of $G(F)$, consisting of all f such that I contains an interval $[\lambda, \mu]$ satisfying 2) for f. The author studies the structure of those subgroups $G(I)$ and proves, in particular, that $G(I)$ is a simple group when I is an open interval. This gives an example of a simple group, which is (simply) ordered and, hence, has no element of finite order except e. *K. Iwasawa* (Cambridge, Mass.).

Zacher, Giovanni. Determinazione dei gruppi finiti strutturalmente omomorfi ad un gruppo d'ordine 8 non ciclico. Rend. Sem. Mat. Univ. Padova 20, 315–328 (1951).

A finite group G is lattice-homomorphic to a non-cyclic group G' of order 8 if and only if $G = R \times S$ with R a group of odd order and either S isomorphic to G' or G' the dihedral group and S the generalized quaternion group of order 16. *P. M. Whitman* (Silver Spring, Md.).

Zacher, Giovanni. Determinazione dei gruppi finiti strutturalmente omomorfi al gruppo generalizzato dei quaternioni e al gruppo abeliano d'ordine 2^s e tipo $(1, 1, \cdots, 1)$. Rend. Sem. Mat. Univ. Padova 20, 346–356 (1951).

Let G' be either a generalized quaternion group or an Abelian group of order 2^s, $s \geqq 3$, of type $(1, 1, \cdots, 1)$. A finite group G is lattice-homomorphic to G' if and only if $G = R \times S$ with R a group of odd order and S isomorphic to G'. *P. M. Whitman* (Silver Spring, Md.).

Zacher, Giovanni. Determinazione dei gruppi finiti strutturalmente omomorfi ad un p-gruppo hamiltoniano finito. Rend. Sem. Mat. Univ. Padova 20, 357–364 (1951).

Let G' be a Hamiltonian p-group, or an Abelian group of order 2^s, $s \geqq 3$, and type $(2, 1, \cdots, 1)$. A finite group G is lattice-homomorphic to G' if and only if $G = R \times S$ with R a group of odd order and S isomorphic to G'. *P. M. Whitman* (Silver Spring, Md.).

Bays, S. Les répartitions imprimitives des n-uples dans le groupe symétrique de degré n. Comment. Math. Helv. 26, 68–77 (1952).

Having previously extended the notion of the imprimitivity of a permutation group of degree n relative to its elements ($i = 1$) to imprimitivity relative to its i-uples ($i > 1$), the author considers the extreme case $i = n$, of which the symmetric group S_n is the only example which is transitive on its n-uples. Corresponding to each sub-group K of S_n, there exist two imprimitive systems of n-uples associated respectively with the partitions of S_n into left and right cosets relative to K. If K is self-conjugate, these two systems coincide. The author studies the various possibilities which arise for subsystems relative to a subgroup H of K. *G. de B. Robinson* (Toronto, Ont.).

Zaremba, S. K. Covering problems concerning Abelian groups. J. London Math. Soc. 27, 242–246 (1952).

Let G be an Abelian group with n base elements, each of order p^k, where p is a prime, and let S denote the set of $\mu = n(p^k - 1) + 1$ distinct powers of the base elements. Under the condition that μ is a power of p^k it is shown that there exists a subset H of G composed of p^{kn}/μ elements such that $G = HS$. Moreover, H cannot be chosen as a subgroup of G unless $k = 1$. This generalizes a previous result of the author [J. London Math. Soc. 26, 71–72 (1951); these Rev. 12, 477]. *L. J. Paige* (Los Angeles, Calif.).

Todd, J. A. On the holomorph of the elementary Abelian group of order 8. J. London Math. Soc. 27, 145–152 (1952).

This paper gives a detailed analysis of the group G of order 1344 which can be defined as the holomorph of the abelian group A of order 8 and type (1, 1, 1). From the table of characters computed by D. E. Littlewood, the author shows that there are two non-isomorphic types of subgroups of order 192 and that G admits two non-equivalent permutation representations of degree 8. *G. de B. Robinson.*

Frame, J. S. The classes and representations of the groups of 27 lines and 28 bitangents. Ann. Mat. Pura Appl. (4) 32, 83–119 (1951).

The group of automorphisms of the 27 lines on a non-singular cubic surface is isomorphic with the symmetry group of a six-dimensional polytope 2_{21} whose 27 vertices have for coordinates the real and imaginary parts of the complex numbers $(0, \omega^\lambda, -\omega^\mu)$, cyclically permuted, where $\omega = e^{2\pi i/3}$ [Coxeter, Amer. J. Math. 62, 457–486 (1940), p. 469; these Rev. 2, 10]. This group, of order $72 \cdot 6!$, contains 36 reflections, corresponding to the interchanges of rows of the 36 double-sixes on the cubic surface. The author obtains an elegant rule for expressing these 36 transformations as matrices. He deduces a complete character table for the group. Five different techniques are used, each yielding some of the characters. He also obtains concise expressions for a basic set of six invariant polynomials [cf. Coxeter,

Duke Math. J. 18, 765–782 (1951), especially pp. 776–779; these Rev. 13, 528].

He then undertakes the analogous investigation of the group of automorphisms of the 28 bitangents of the general plane quartic curve. This group, of order $4 \cdot 9!$, is closely related to the symmetry group of a seven-dimensional polytope 3_{21}, whose 56 vertices belong to seven concentric cubes, lying in distinct 3-spaces every two of which have a common line [Coxeter, Proc. London Math. Soc. (2) 34, 126–189 (1932), p. 186]. Here the task is still more formidable, but the character table is again successfully completed. To illustrate its usefulness, he describes the subgroups of indices 28, 36, 56, 63, 72, 120, 126, 135, and proves that these are the only large subgroups (index <160).

H. S. M. Coxeter (Toronto, Ont.).

Zappa, Guido. Sulle p-catene dei gruppi p-risolubili. Giorn. Mat. Battaglini (4) 3(79), 121–126 (1950).

The author extends a theorem of Ore [Duke Math. J. 3, 149–174 (1937)] to p-soluble groups in the sense of Čunihin [C. R. (Doklady) Acad. Sci. URSS 55, 477–480 (1947); these Rev. 9, 6; see also these Rev. 11, 495]. (Actually Čunihin calls a group p-soluble, if it has a composition series with factors which are either p or prime to p. The author postulates something apparently stronger, viz. the existence of a composition series through a subgroup whose order is prime to p and index a power of p. That such a series exists for every p-soluble group can be deduced from Schur's well known theorem.) Let then G be a p-soluble group of order $p^a m$, $\alpha>0$, $(m, p)=1$. Let $G=H_0 \supset H_1 \supset \cdots \supset H_r$ be any complete p-chain of subgroups, i.e. H_{i+1} maximal subgroup of index a power of p in H_i, and H_r of order m. Further let $G=N_0 \supset N_1 \supset \cdots \supset N_t$ be any complete normal p-chain of subgroups, i.e. N_{j+1} maximal normal subgroup of index a power of p in N_j, and N_t of order m. Theorem: The p-indices of the second chain can be subdivided into s blocks so that the indices of the first chain and the blocks of the second chain are in 1-1-relation with the product of the indices within a block equal to the corresponding index in the first chain.

K. A. Hirsch (London).

Čunihin, S. A. On weakening the conditions in theorems of Sylow type. Doklady Akad. Nauk SSSR (N.S.) 83, 663–665 (1952). (Russian)

Let the terminology and notation be as described earlier [Mat. Sbornik N.S. 25(67), 321–346 (1949); same Doklady (N.S.) 66, 165–168 (1949); 69, 735–737 (1949); 73, 29–32 (1950); 77, 973–975 (1951); these Rev. 11, 495; 10, 678; 11, 321; 12, 156, 800]. The notation $\mathfrak{G}//\mathfrak{G}_1$ is now used to denote the factor-structure previously written $[\mathfrak{G}, \mathfrak{G}_1]$. In the definition of a factor-structure of type II-1, if "normal series" is replaced by "II-permutable series", we obtain the definition of a factor-structure of type II*1. In the definition of a factor-structure of type II-2, if "II-1" is replaced by "II*1", we obtain the definition of a factor-structure of type II*2. A factor structure $\mathfrak{G}//\mathfrak{E}$ is of type II*1 if and only if it is of type II-1. The corresponding result however need not be valid for a factor-structure $\mathfrak{G}//\mathfrak{G}_1$. A necessary and sufficient condition that $\mathfrak{G}//\mathfrak{E}$ be of type II-1 is that $\mathfrak{G}$ possess a II-permutable series every factor-structure of which is of type II*1. A necessary and sufficient condition that $\mathfrak{G}//\mathfrak{E}$ be of type II-2 is that $\mathfrak{G}$ possesses a II-semi-invariant series every factor-structure of which is of type II*2.

R. A. Good (College Park, Md.).

Kazačkov, B. V. On a local theorem in the theory of groups. Doklady Akad. Nauk SSSR (N.S.) 83, 525–528 (1952). (Russian)

A collection L of subgroups of a group $\mathfrak{G}$ is called a local system of subgroups provided every element of $\mathfrak{G}$ belongs to a member of L and provided every two members of L are contained in a member of L. If Π is a non-empty set of primes, $\mathfrak{G}$ is said to possess the finitely Π-conjugate property provided either $\mathfrak{G}$ contains no finite class of conjugate Sylow Π-subgroups or else, if such a class exists, then all Sylow Π-subgroups of $\mathfrak{G}$ are conjugate. If $\mathfrak{G}$ has a local system of subgroups every member of which has the finitely Π-conjugate property, then $\mathfrak{G}$ itself possesses the finitely Π-conjugate property. This generalizes certain earlier results by the author [Doklady Akad. Nauk SSSR (N.S.) 80, 5–7 (1951); these Rev. 13, 431].

R. A. Good (College Park, Md.).

Hughes, N. J. S. The unique decomposition of regular ω-linear mappings as direct products. Proc. Amer. Math. Soc. 3, 359–362 (1952).

In a previous paper [same Proc. 2, 742–747 (1951); these Rev. 13, 528] the author has defined regular bilinear (n-linear) mappings on an Abelian group H (a set of Abelian groups H^i for $i<n$) to a group K and has proved a uniqueness theorem for direct decompositions of such mappings. In the present note he extends his definitions by transfinite induction to α-linear mappings where α is an infinite ordinal of the second number class. He requires a set of groups H^i for $i<\alpha$ (which turn out to be Abelian), a group K, and mappings f of the H^i to K which in addition to the previous linearity properties must have certain limit-properties. Again he defines decomposability of a mapping and proves a refinement theorem from which the uniqueness of the decomposition as a product of indecomposable mappings follows, provided that one such decomposition exists.

K. A. Hirsch (London).

Yamabe, Hidehiko. A condition for an abelian group to be a free abelian group with a finite basis. Proc. Japan Acad. 27, 205–207 (1951). Read the corresponding note in 5·1140

Let G be a countable Abelian group, written additively. The author calls an integer-valued function $f(x, y)$, defined on $G \times G$, bilinear, if for any $x_i, y_j \in G$, $i, j=1, 2$, $f(x_1+x_2, y_1+y_2)=\sum_{i,j} f(x_i, y_j)$. He asserts and proves the theorem that G is free Abelian, if a bilinear integer-valued function exists which vanishes for the identity element of $G \times G$ only. The reviewer confesses that he does not understand the proof and that in his opinion the theorem is true, but empty, because bilinear functions with the required property do not exist. Indeed for any $x \in G$ we have $f(x+0, 0+0)=f(x, 0)+f(x, 0)+f(0, 0)+f(0, 0)$, and hence, if $f(0, 0)=0$, then $f(x, 0)=0$ for all $x \in G$. For a different definition of bilinear mappings between Abelian groups see N. J. S. Hughes [Proc. Amer. Math. Soc. 2, 742–747 (1951); these Rev. 13, 528].

K. A. Hirsch (London).

Meier-Wunderli, H. Note on a basis of P. Hall for the higher commutators in free groups. Comment. Math. Helv. 26, 1–5 (1952).

The reviewer constructed an explicit basis for free Lie rings and for higher commutators in free groups [Proc. Amer. Math. Soc. 1, 575–581 (1950); these Rev. 12, 388]. The reviewer noted that for free groups these were the commutators arising in Philip Hall's collecting process [Proc. London Math. Soc. (2) 36, 29–95 (1933), p. 67].

The present paper has as its object a proof, not depending on Lie rings, of the basis theorem for commutators in free groups and the formula due to Witt [J. Reine Angew. Math. 177, 152–160 (1937)] which gives for each weight the number of elements in the basis. According to the author there is an unpublished proof of the basis theorem due to Philip Hall depending on the Witt formula. The proofs given in this paper of the basis theorem and the Witt formula are ingenious in conception, but both are incomplete. In the proof of Lemma 4 it is necessary to show that the mapping $P_j \to P_j P_k$ is a homomorphism before it can be represented as induced by transformation. And in the proof of the Witt formula it is not obvious to the reviewer that the bracketing described can be completed to yield a standard commutator or a standard commutator repeated. *Marshall Hall.*

Tartakovskiĭ, V. A. On primitive composition. Mat. Sbornik N.S. **30(72)**, 39–52 (1952). (Russian)

This paper is a sequel to the author's previous work on the word problem in groups [Mat. Sbornik N.S. **25(67)**, 3–50, 251–274 (1949); see also Amer. Math. Soc. Translation no. 60 (1952); these Rev. **11**, 493; **13**, 528]. A knowledge of the relevant definitions, notations, and theorems is indispensable for the understanding of the present paper and review. One of the major inconveniences in the author's treatment of the word problem (see point (5) of the review cited) was the fact that if one of the generators entering into a cyclic word has infinite (formal) order then there is an infinity of formally unequal ways of splitting the word and hence an infinity of distinct left fundamental rays in case of a Dyck word. To overcome this the author was compelled to introduce words in which the exponents of the generators were not integers but linear functions of integer-valued arguments with integer coefficients. The idea of the present paper is to obviate these complications by excluding all compositions which arise from splits within a link of one of the factors and admitting only compositions with splits at the boundary between two links. Such a split is called by the author "primitive". The restriction to primitive compositions might appear at first sight so severe that it ruins the proofs (conjugacy problem!). However, the author successfully carries through the modifications in definitions (primitive product, primitive extinction etc.) and proofs and obtains as his principal results the following two theorems which replace the corresponding theorems of the first paper cited. Theorem 1′: The zone of extinction of any system of cyclic Dyck words coincides with the zone of primitive extinction of these words. Theorem 2′: The domain of extinction of a group G given by generators S_i, $i=1, 2, \cdots, m$, and defining relations $f_j(S_i)=1$, $j=1, 2, \cdots, n$, (possibly together with formal order relations for some of the generators) is the sum of the stars of extinction of all its simplest Dyck words. Here a simplest Dyck word is one which is reduced and freely equal to a simple primitive product of the basic factors $f_i^{\epsilon_i}$ ($\epsilon_i=\pm 1$). These theorems represent an enormous simplification in the actual algorithm which leads to a decision about the equality of two words in a group which yields to the author's method. It should be noted that the refinement of his proof which gave the improvement "$k>6$" instead of "$k>8$" [see Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 483–494 (1949); these Rev. **11**, 493; **13**, 528] cannot be carried out with primitive compositions.

K. A. Hirsch (London).

Lyndon, Roger C. Cohomology theory of groups with a single defining relation. Ann. of Math. (2) **52**, 650–665 (1950).

Für eine Gruppe G, die durch endlich viele Erzeugende und eine einzige Relation R gegeben ist, wird (1) das Identitätsproblem gelöst und (2) die Bestimmung der Cohomologiegruppen $H^p(G, K)$ bezüglich eines G-Modulus K als Koeffizientenbereich durchgeführt. Zu (1): Identitäten zwischen den Transformierten von Relationen, welche stets bestehen, wurden von Peiffer [Math. Ann. **121**, 67–99 (1949); diese Rev. **11**, 322] untersucht. In der vorliegenden Arbeit wird gezeigt: (a) Ist R eine echte Potenz Q^q, $q>1$ (wobei q maximal sei), so sind alle Identitäten Folgen der Peifferschen und der Identitäten $Q^sRQ^{-s}R^{-1}=1$. (b) Ist R keine echte Potenz, so folgen alle Identitäten aus den Peifferschen. Zu (2): Die Gruppen $H^p(G, K)$, $p\geqq 2$, ergeben sich aus (1) auf Grund von Beziehungen zwischen den Identitäten und den Inzidenzmatrizen eines abstrakten azyklischen Komplexes, in welchem G operiert. Besonders einfach ist das Ergebnis für $p\geqq 3$: Im Falle (b) sind die $H^p(G, K)=0$; im Falle (a), wenn im Gruppenring von G $s=1+Q+\cdots+Q^{q-1}$ und $t=1-Q$ gesetzt und unter ${}_sK$ bezw. ${}_tK$ der Teilmodul von K verstanden wird, der durch s bezw. t annulliert wird, ist, für gerades $p\geqq 4$, $H^p(G, K)={}_sK/sK$, für ungerades $p\geqq 3$ $H^p(G, K)={}_tK/tK$. Die Methoden der Herleitung gehören der Theorie der freien Gruppen an (Freiheitssatz, freie Produkte) und benützen als übersichtliches formales Hilfsmittel die von R. H. Fox [Bull. Amer. Math. Soc. **55**, 276 (abstract 115), 512 (abstract 228) (1949)] eingeführte Derivation in freien Gruppen. Azyklische Komplexe und algebraische Cohomologietheorie werden in enger Anlehnung an die Reidemeistersche Theorie der Homotopieketten dargestellt und benützt. *B. Eckmann.*

Faddeev, D. K. On the theory of homology in groups. Izvestiya Akad. Nauk SSSR. Ser. Mat. **16**, 17–22 (1952). (Russian)

The author studies a relation between the cohomology groups of a group G, and those of a subgroup H. Let G operate on the right of the additive group A, and let i be the additive group of functions from the set $\{\rho\}$ of right cosets of H in G to A, with G operating by $f^x(\rho)=[f(\rho x^{-1})]^x$. Theorem: $H_n(G, i)\approx H_n(H, A)$. The isomorphism is derived from the chain map η (the y_i run through H; $\rho_0=H$): $\eta F(y_1, \cdots, y_n)=F(y_1, \cdots, y_n)(\rho_0)$. The proof utilizes a number of other groups: $C_n(G, H, A)$ is the group of H-homogeneous cochains (on G with values in A):

$$f(x_1y, \cdots, x_ny)=f^y(x_1, \cdots, x_n)$$

for $y\in H$; there is an explicit isomorphism between $C_n(G, i)$ (non-homogeneous cochains) and $C_{n+1}(G, H, A)$, commuting with coboundary. It remains to prove that

$$H_{n+1}(G, H, A)\approx H_{n+1}(H, H, A),$$

the isomorphism induced by restriction of the functions to H. This is shown by induction; an important role is played by the theorem that for $n\geqq 2$ all $H_n(G, H, b)=0$, where b is the group of all functions from H to A, with H operating by $f^{y_1}(y)=[f(yy_1^{-1})]^{y_1}$. Reviewer's note: An interpretation of the author's result can be obtained by noting that the Eilenberg-MacLane space of H is a covering space of that of G.

H. Samelson (Ann Arbor, Mich.).

Foulkes, H. O. Monomial symmetric functions, S-functions and group characters. Proc. London Math. Soc. (3) 2, 45–59 (1952).

The problem of expressing a monomial symmetric function in terms of Schur functions and vice versa is of interest in several contexts. Kostka's tables, computed in 1882, are significant in this connection and the author shows how these may be expressed in matrix form. A method of D. E. Littlewood for solving the problem is here extended and the results applied to the calculation of the characters of certain classes of elements of the symmetric group.

G. de B. Robinson (Toronto, Ont.).

Murnaghan, F. D. The element of volume of the rotation group. Proc. Nat. Acad. Sci. U. S. A. 38, 69–73 (1952).

The explicit formula the author has given for the volume element of the n-dimensional rotation group in Eulerian parameters [same Proc. 36, 670–672 (1950); these Rev. 12, 352] is modified in such a way that the element is split into a class factor and an in-class factor. *H. Freudenthal.*

Murnaghan, F. D. On a convenient system of parameters for the unitary group. Proc. Nat. Acad. Sci. U. S. A. 38, 127–129 (1952).

Any $n\times n$ unitary matrix U_n may be written $U_n = VAV^{-1}$, where A is a diagonal matrix $(e^{i\alpha_1}, \cdots, e^{i\alpha_n})$ and

$$V = U_{12}U_{23}\cdots U_{n-1,n}U_{12}\cdots U_{12}$$

in which U_{ij} has 1's in the leading diagonal and a square matrix

$$\begin{pmatrix} \cos\theta e^{i\tau} & -\sin\theta \\ \sin\theta & \cos\theta e^{-i\tau} \end{pmatrix}$$

at the intersections of the i, j rows and columns with zeros elsewhere. *G. de B. Robinson* (Toronto, Ont.).

Dieudonné, Jean. On the orthogonal groups over an algebraic number field. Proc. London Math. Soc. (3) 2, 245–256 (1952).

Let E be an n-dimensional vector space over a field K, f a non-degenerate symmetric bilinear form on E, and let $O_n(K, f)$ denote the corresponding orthogonal group over E and $\Omega_n(K, f)$ the commutator subgroup of $O_n(K, f)$. In the present paper, the author considers the group $O_n(K, f)$ over a finite algebraic number field K and gives generalizations of the results in his previous papers, where $K = Q$ was the rational field [cf. Ann. of Math. 54, 85–93 (1951); Amer. J. Math. 73, 940–948 (1951); these Rev. 13, 531]. In particular, it is proved that, if K has $r_1 \geqq 1$ real conjugates and the index ν of f is 0, then the projective group $P\Omega_n(K, f)$ is simple for $n \geqq 3\cdot 2^{r_1}$. Note that any form f over a purely imaginary field K has an index $\nu \geqq 1$ and that, whatever the field K is and $n \geqq 5$, $P\Omega_n(K, f)$ is always simple when f has an index $\nu \geqq 1$ [cf. the author's book, Sur les groupes classiques, Hermann, Paris, 1948, p. 29; these Rev. 9, 494].

The proof of the above theorem is based upon the following proposition: If the space E contains a subspace F of dimension $\geqq 6$, such that $f(x, x)$ is totally positive for every $x \neq 0$ in F, then $P\Omega_n(K, f)$ is simple. It is also proved that $O_n(K, f)/\Omega_n(K, f)$ is an infinite group for any algebraic number field K, any form f and $n \geqq 2$, and that, if ρ is a totally positive number in K and f_1 a totally positive hermitean form in an n-dimensional space E_1 over $K_1 = K(\sqrt{-\rho})$, then the commutator subgroup of the projective unitary group $PU_n(K_1, f_1)$ over E_1 is simple for $n \geqq 5$.

K. Iwasawa (Cambridge, Mass.).

Pettis, B. J. A note on everywhere dense subgroups. Proc. Amer. Math. Soc. 3, 322–326 (1952).

Let X be a second category topological group. The author conjectures that each proper subgroup of X is contained in a certain second category proper subgroup of X, and verifies it for the following two cases: (i) X is abelian and, for each positive integer r, the power map $x \to x^r$ carries non-null open sets into somewhere dense sets; (ii) X is an abelian Lie group. Also proved is that an everywhere dense proper subgroup of a second category, non-discrete, Hausdorff group X must lie in some non-countable proper subgroup of X (this is a sharpening of a theorem of the reviewer [Proc. Amer. Math. Soc. 1, 18–19 (1950); these Rev. 11, 416]). As the author points out, most of the results can be generalised to semi-groups with unit. *H. C. Wang.*

Kawada, Yukiyosi. Corrections to my paper: On some properties of a topological group. J. Math. Soc. Japan 3, 349 (1951).

See same J. 1, 203–211 (1950); these Rev. 12, 318.

Hu, Sze-tsen. Cohomology rings of compact connected groups and their homogeneous spaces. Ann. of Math. (2) 55, 391–419 (1952).

If $X = G/G_*$ is the homogeneous space of a compact connected Lie group G by a connected subgroup G_*, the cohomology ring $H(X)$ with real coefficients depends only on the Lie algebra L of G and the subalgebra L_* corresponding to G_* [Chevalley and Eilenberg, Trans. Amer. Math. Soc. 63, 85–124 (1948); these Rev. 9, 567]. In the present paper the author generalizes this result to the case where G is an arbitrary compact connected group and the Lie algebra L has been replaced by the concept of local group. If V denotes a local group and V_* is a local subgroup of V, a cohomology ring $H(V, V_*)$ with real coefficients is defined. The main result asserts that if X is the homogeneous space of a compact connected group G by a connected subgroup G_* and if V denotes any open neighborhood of the identity in G and $V_* = V \cap G_*$, then the cohomology ring $H(X)$ of X with real coefficients is isomorphic to $H(V, V_*)$. In the above $H(X)$ denotes the Alexander-Wallace cohomology ring [Spanier, same Ann. 49, 407–427 (1948); these. Rev. 9, 523]. It is an immediate consequence of this theorem that locally isomorphic compact connected groups have isomorphic cohomology rings. The proof depends, firstly, on the fact that the Alexander-Wallace cohomology ring with real coefficients is isomorphic to a corresponding ring based on continuous cochains. Secondly, it is shown that if a compact connected group acts on a space, the continuous cohomology ring of the space is isomorphic to the invariant cohomology ring (ring based on invariant cochains). If the group acts transitively, it is proved that the invariant cohomology ring is a local property. Finally the ring $H(V, V_*)$ is defined and the main result established. *E. Spanier.*

Harish-Chandra. Plancherel formula for the 2×2 real unimodular group. Proc. Nat. Acad. Sci. U. S. A. 38, 337–342 (1952).

In an earlier note [same Proc. 37, 813–818 (1951); these Rev. 13, 533] the author has given an explicit Plancherel formula for connected complex semi-simple Lie groups. In the present note he observes that the corresponding problem for real groups is much more difficult and begins an attack upon it by solving it for one particular such group; the group G of all 2×2 real matrices of determinant one. He

describes two one-parameter families τ_λ^+ and τ_λ^- and two discrete families σ_n^+ and σ_n^- of unitary representations of G and then sketches a proof of the formula:

$$8\pi f(1)=\frac{1}{2}\int_{-\infty}^{\infty}\lambda\tanh\pi\lambda T_\lambda^+(f)d\lambda+\frac{1}{2}\int_{-\infty}^{\infty}\lambda\coth\pi\lambda T_\lambda^-(f)d\lambda$$
$$+\sum_{n\geq 0}(n+\tfrac{1}{2})S_n^+(f)+\sum_{n\geq 0}(n+1)S_{n+1}^-(f)$$

where

$$T_\lambda^\pm(f)=\text{Trace}\left(\int f(x)\tau_\lambda^\pm(x)dx\right),$$

$$S_n^\pm(f)=\text{Trace}\left(\int f(x)\sigma_n^\pm(x)dx\right)$$

and f is any infinitely differentiable function on G with compact support. Connections with the work of Bargmann [Ann. of Math. (2) 48, 568–640 (1947); these Rev. 9, 133] on the representations of G are briefly indicated.

G. W. Mackey (Cambridge, Mass).

Godement, Roger. Une généralisation du théorème de la moyenne pour les fonctions harmoniques. C. R. Acad. Sci. Paris 234, 2137–2139 (1952).

Let G be a connected Lie group and let K be a compact subgroup of G. The author defines a notion of harmonic function in terms of differential operators for functions defined on open subsets of the homogeneous space G/K and shows that a function is harmonic if and only if its value at every point coincides with certain mean values. Specifically, let U denote the set of all differential operators on G which annihilate constants, commute with all left translations and are invariant under all right translations by members of K. The function F is said to be harmonic if it is defined in an open subset O of G, if both F and O are invariant under right translation by members of K and if F is carried into zero by all operators in U. The mean-value theorem asserts that F is harmonic if and only if $F(g)=\int_K F(gkx)dk$ for all g in O and all x in G sufficiently close to the identity. This theorem reduces to the classical one when G is the group of rigid motions in R^n and K is the subgroup of rotations. *G. W. Mackey.*

Murakami, Shingo, and Gotô, Morikuri. On the inner automorphisms of a compact group. Nagoya Math. J. 4, 119–123 (1952).

The following theorem is proved: If an automorphism of a compact connected group G transforms every element of G into a conjugate element, it is an inner automorphism. By Peter and Weyl's theorem, this is equivalent to the following: Let τ be an automorphism of a compact connected group; if for every representation D: $x\to D(x)$ of G, the representation D^τ: $x\to D(x^\tau)$ is equivalent to D, τ is inner. According to the authors, the latter result had been obtained by M. Abe [On Lie algebras with commutative radical (in Japanese), Zenkoku Sijô Sûgaku Danwakai 226, 560–574 (1941)] by making use of the classification of simple Lie algebras, which is avoided here. Abe's theorem is used to give, in the connected case, a new simple proof of the following theorem, due to Iwasawa [Ann. of Math. (2) 50, 507–558 (1949); these Rev. 10, 679]: Let G be a compact group, $A(G)$ the group of all automorphisms of G with the compact-open topology, and $I(G)$ the subgroup of all inner automorphisms; then, the quotient group $A(G)/I(G)$ is totally disconnected. The proof is essentially a generalization of Iwasawa's proof for the abelian case [loc. cit., p. 514].

J. L. Tits (Princeton, N. J.).

Kuranishi, Masatake. On one-parameter subgroups in finite dimensional locally compact group with no small subgroups. Nagoya Math. J. 4, 89–96 (1952).

Theorem: A finite-dimensional locally compact group without small subgroups has a canonical family of one-parameter subgroups. This result generalizes that of Chevalley [Proc. Amer. Math. Soc. 2, 122–125 (1951); these Rev. 13, 12] and Yamabe [Osaka Math. J. 3, 77–82 (1951); these Rev. 13, 13] by dropping the requirement that the group be locally connected. The proof depends on the theorem of Montgomery which asserts that an n-dimensional group contains an n-dimensional locally connected local normal subgroup whose local quotient group is zero-dimensional. *A. Gleason* (Washington, D. C.).

Montgomery, Deane, and Zippin, Leo. Small subgroups of finite-dimensional groups. Proc. Nat. Acad. Sci. U. S. A. 38, 440–442 (1952).

The authors outline the proof of a theorem which, combined with a recent result of Gleason, gives a complete solution of the outstanding Hilbert's fifth problem on continuous groups. The theorem is as follows: If G is a separable, metric, locally compact, finite-dimensional, connected and locally connected topological group, and if all proper subgroups of G are generalized Lie groups, then G contains an invariant closed generalized Lie subgroup H such that G/H is finite-dimensional and has no small subgroups. The essential steps in proving this theorem are the following propositions. 1) If G satisfies the assumptions of the theorem, G contains an invariant generalized Lie subgroup H such that G/H satisfies the same assumptions and in addition has only the identity in its center. 2) The group G/H of 1) contains no infinite compact zero-dimensional group which commutes element-wise with a subgroup of positive dimension. As a corollary G/H can contain no infinite compact abelian zero-dimensional subgroup. 3) The group G/H of 2) has no small finite subgroups. As a corollary, it has no small subgroups at all.

The above theorem, together with a theorem of Gleason that a finite-dimensional group without small subgroups is a Lie group, gives an important result that every finite-dimensional (separable, metric, locally compact) group is a generalized Lie group. It then follows immediately that every locally euclidean group is a Lie group, namely, the solution of Hilbert's fifth problem. *K. Iwasawa.*

NUMBER THEORY

*Sierpiński, Wacław. Teoria liczb. [Theory of numbers.] 3rd ed. Monografie Matematyczne, Tom XIX. Warszawa, Wrocław, 1950. vi+544 pp.

This is a new edition of a book first published in 1914, but with some modifications and additions. It is an exposition of the elementary theory of numbers together with the most elementary parts (as Tchebyscheff's theorem on density of prime numbers, Gauss's theory of binary quadratic forms, elementary theory of quadratic number fields) of its more advanced fields (analytic theory of numbers, diophantine equations, quadratic forms, algebraic numbers), serving as introduction to a further study of these fields. The pur-

pose of the author is to give, at the same time, a text-book in number theory for beginning students and an elementary treatise initiating secondary school teachers (and even non-mathematicians interested in mathematics) into this theory. Thus, though elementary, this book is written, within the limits of its subject, with a rather encyclopedic scope. In addition to the proved results, many more special or more difficult ones are quoted without proof. The book contains also many problems and exercises.

Table of contents: (1) Divisibility of numbers and decomposition into prime factors; (2) Indeterminate equations of first degree; (3) Fundamental properties of congruences, congruences of first degree with prime modulus; (4) Theorems of Wilson, Euler and Fermat; theorems on decomposition into a sum of squares; (5) Number and sum of factors, perfect numbers, summation formulas; (6) Möbius' function, Gauss' function, the relation $F(n)=\sum_{d|n}f(d)$ and its inversion; (7) Density of distribution of prime numbers in the sequence of natural numbers; (8) Euler's theorem, Lagrange's theorem, primitive residues and indices; (9) Systematic developments with an arbitrary base of numeration; (10) Pythagoras' equation and its generalizations; (11) Pell's equation; (12) Continued fractions; (13) Theory of congruences of first and second degree; (14) Theory of Legendre's symbol and of Jacobi's symbol; (15) Sketch of the theory of quadratic forms; (16) Theory of complex integers; (17) Introduction to the theory of numerical fields; (18) Introduction to the theory of ideals; (19) Fermat's great theorem for exponents 5 and 7; (20) Complements to various chapters. *M. Krasner.*

Sierpiński, W. Sur l'existence des nombres premiers avec une suite arbitraire de chiffres initiaux. Matematiche, Catania **6**, 135–137 (1951).

Let $a_1, a_2, \cdots, a_m$ be any ordered sequence of m decimal digits. The author shows by a simple application of the prime number theorem that there are infinitely many primes whose first m digits are the given a's. Also, by Dirichlet's theorem, there are infinitely many primes whose last digits are the a's, provided a_m is prime to ten. *D. H. Lehmer.*

Greenwood, Thomas. Introduction aux nombres polygones. Rev. Trimest. Canad. **38**, 58–75 (1952).

Jarden, Dov. Factorization-formulae for numbers of Fibonacci's sequence decreased or increased by a unity. Riveon Lematematika **5**, 55–58 (1952). (Hebrew)

The author establishes the identities

$$\left.\begin{aligned} U_{4n+k}-1&=U_{2n+\epsilon_k}V_{2n+k-\epsilon_k}\\ U_{4n+k}+1&=V_{2n+\epsilon_k}U_{2n+k-\epsilon_k}\end{aligned}\right\}k=0,1,2,3;$$

$$\epsilon_0=1,\quad \epsilon_1=\epsilon_2=0,\quad \epsilon_3=2;$$

here U_n is the nth Fibonacci number and $V_n=U_{2n}/U_n$. Similar identities are obtained for $V_{4n+k}\pm1$; and the results are applied to the proof of certain divisibility properties of the U_n and V_n. Factorization tables are given of $U_n\pm1$ and $V_n\pm1$ for $0\leqq n\leqq 61$. *E. G. Straus* (Los Angeles, Calif.).

Rosati, Luigi Antonio. Risoluzione di un sistema diofanteo. Boll. Un. Mat. Ital. (3) **7**, 69–70 (1952).

Boldyreff, A. W. Methods of solution of Diophantine equations by elementary means. Bol. Mat. **24**, 31–40 (1951).

LeVeque, Wm. J. On the equation $a^x-b^y=1$. Amer. J. Math. **74**, 325–331 (1952).

For fixed positive integers a, b the title equation has at most one integer solution x, y except that $a=3$, $b=2$ has $3-2=3^2-2^3=1$. The proof for $2\mid a$ is simple, that for $2\nmid a$ is elaborate. The solution, if it exists, is specified entirely by certain congruences. As a corollary it is shown that the only identity $\sum^n k^s=(\sum^n k^t)^r$ with integer s, r, t is $s=3$, $t=1$, $r=2$. *J. W. S. Cassels* (Cambridge, England).

Mordell, L. J. On cubic equations $z^2=f(x, y)$ with an infinity of integer solutions. Proc. Amer. Math. Soc. **3**, 210–217 (1952).

The author continues his investigations [J. London Math. Soc. **17**, 199–203 (1942); Ann. Mat. Pura Appl. (4) **29**, 301–305 (1949); these Rev. **4**, 265; **6**, 334; **11**, 714] on certain classes of cubic Diophantine equations. It is proved that the equation

$$z^2=p^3+\lambda x+ax^2+bxy+2py^2+Ax^3+Bx^2y+Cxy^2+Dy^3,$$

with $p\neq0$, has infinitely many solutions if $p^3(b^2-8ap)+2p\lambda^2$ is positive and not a square. Also the same equation with quadratic terms in x and y deleted, $\lambda p\neq0$, has infinitely many solutions if the congruence

$$2\lambda^3+2p(8Ap^3+4Bp^2t+2Cpt^2+Dt^3)\equiv0 \pmod{\lambda^4}$$

is solvable for t. The equation $z^2=p^3+\lambda x+ax^2+bxy+cy^2$ with $\lambda p\neq0$ has infinitely many solutions if

$$(9a+3bt+ct^2)p^4-3\lambda^2p\equiv0 \pmod{\lambda^3}$$

is solvable. Note the unconditional results in these last two cases if $\lambda=1$. Finally, let L_1, M_1 be homogeneous polynomials in x, y, z of the first degree, L_2, M_2 similarly of the second degree, integral coefficients throughout. Then $L_1+L_2+M_1^3+L_1M_2=0$ has infinitely many solutions if $L_1=0$, $M_1=1$ are solvable in integers. It is conjectured in general that if the cubic equation $g(x, y, z)=0$ has one integral solution, then there are infinitely many solutions if $g(x, y, z)-a$ is irreducible for all a. *I. Niven.*

Dénes, Peter. An extension of Legendre's criterion in connection with the first case of Fermat's last theorem. Publ. Math. Debrecen **2**, 115–120 (1951).

By Legendre's criterion is meant the theorem that if both l and $2l+1$ are primes, then the equation (*) $x^l+y^l+z^l=0$ has no solutions x, y, z, prime to l. The present paper extends this result to cover 35 different cases as follows: Let u be any even number less than 104 not divisible by 6. If both l and $ul+1$ are primes, then (*) has no solutions prime to l. This theorem is also proved for $u=110$. The results follow from the fact that this first case of Fermat's last theorem is proved for $l<253747899$ [D. H. and Emma Lehmer, Bull. Amer. Math. Soc. **47**, 139–142 (1941); these Rev. **2**, 250]. The following lemma is used: Let u be a positive integer and ρ a primitive uth root of unity. Let $N_u(z)$ denote the norm of a number z in the cyclotomic field generated by ρ. If both l and $p=ul+1$ are primes and if $N_u(1+\rho^a+\rho^b)$ is divisible by p for no choice of the integers a and b, then (*) has no solution prime to l. If u is not divisible by 6, then $N_u(1+\rho^a+\rho^b)$ cannot vanish. Furthermore, for a fixed u, if $N_u(1+\rho^a+\rho^b)$ does not exceed p, then the hypothesis of the lemma is satisfied. The author finds upper bounds for $N_u(1+\rho^a+\rho^b)$ which, for the u's mentioned above, are in each case less than $253747899u$. *D. H. Lehmer.*

Oblath, Richard. Eine Bemerkung über Produkte aufeinander folgender Zahlen. J. Indian Math. Soc. (N.S.) **15** (1951), 135–139 (1952).

It is proved that the product $P_k(x)=x(x+1)\cdots(x+k-1)$ of k consecutive integers cannot be a perfect nth power y^n for $2\leqq k\leqq 17$, $y\neq 0$, $n>1$. The author had treated some cases previously [Revista Mat. Hisp.-Amer. (4) **2**, 190–210, 253–270 (1942); these Rev. **5**, 90], and attributes part of his method to Szekeres. The proof employs Pillai's theorem [Proc. Indian Acad. Sci., Sect. A. **11**, 6–12 (1940); these Rev. **1**, 199] that in every set of 16 or fewer consecutive integers there is one which is prime to all the others, and the exceptions to this for 17 integers noted by Pillai. The result overlaps various earlier contributions to the conjecture that $P_k(x)$ cannot be an nth power by Erdös [J. London Math. Soc. **14**, 194–198, 245–249 (1939); these Rev. **1**, 4, 39], Johnson [Amer. Math. Monthly **47**, 280–289 (1940); these Rev. **1**, 291] and Pillai [Proc. Indian Acad. Sci., Sect. A. **11**, 73–80 (1940); these Rev. **1**, 291]. *I. Niven.*

Cheo, Luther. A remark on the $\alpha+\beta$ theorem. Proc. Amer. Math. Soc. **3**, 175–177 (1952).

Lepson in a recent paper [same Proc. **1**, 592–594 (1950); these Rev. **12**, 392] proved the following theorem: Let α and β be two arbitrary real numbers satisfying $\alpha>0$, $\beta>0$, $\alpha+\beta\leqq 1$. Then, there exist two sets of positive integers A and B of Schnirelmann density α resp. β so that the Schnirelmann density of $A+B$ equals $\alpha+\beta$. The author now proves the following stronger result: Let α, β, γ be three arbitrary real numbers satisfying $\alpha>0$, $\beta>0$, $\alpha+\beta\leqq\gamma\leqq 1$. Then, there exist two sets of positive integers A and B of Schnirelmann density α resp. β so that the Schnirelmann density of $A+B$ is γ. *P. Erdös* (Los Angeles, Calif.).

Cohen, Eckford. Arithmetic functions of polynomials. Proc. Amer. Math. Soc. **3**, 352–358 (1952).

The paper is concerned with arithmetic functions defined over the ring $GF[p^n, x]$. In place of the functions ϵ_{qH} defined by the reviewer [Duke Math. J. **14**, 1121–1137 (1947); these Rev. **9**, 337], the writer shows that a certain simpler set ϵ_E defined by $\epsilon_E(A)=\epsilon(ZA, R)$, where R is a fixed polynomial in $GF[p^n, x]$, can be used to advantage in certain problems. A number of applications are given. In particular the author determines the number of solutions of

$$F=\alpha_1P_1X_1Y_1+\cdots+\alpha_sP_sX_sY_s \quad (\alpha_i\epsilon GF(p^n)),$$

where the P_i are irreducible and the unknown X_i, Y_i satisfy certain restrictions. The author points out that the method of the present paper can be applied in the rational case; this is discussed in the paper reviewed below. *L. Carlitz.*

Cohen, Eckford. Rings of arithmetic functions. Duke Math. J. **19**, 115–129 (1952).

Given a field F of characteristic 0 containing the rth roots of unity for a positive integer r. Then a single-valued function $f(a)$ in F is called (r, F) arithmetic if $f(a)=f(a')$ for $a\equiv a' \pmod r$. It is proved that any (r, F) arithmetic function $f(a)$ can be expressed in the form $\sum_{s=0}^{r-1}a_s\epsilon_s(a)$ with $\epsilon_s(a)=e^{2\pi ias/r}$ and a_s in F. Thus the (r, F) arithmetic functions are generalizations of common Gauss sums. Some algebraic properties of the set of (r, F) arithmetic functions are given. *H. Bergström* (Göteborg).

Cohen, Eckford. Sur les fonctions arithmétiques relatives aux corps algébriques. C. R. Acad. Sci. Paris **234**, 787–788 (1952).

The definitions and theorems in the paper reviewed above are here generalized from rational fields to algebraic fields. *H. Bergström* (Göteborg).

Bochner, S. Remarks on Gaussian sums and Tauberian theorems. J. Indian Math. Soc. (N.S.) **15** (1951), 97–104 (1952).

The author points out that Kronecker's derivation of the reciprocity formula for certain Gaussian sums from the modular relation

$$\tfrac{1}{2}+\sum_{n=1}^{\infty}e^{-\pi n^2/x}=x^{1/2}\left\{\tfrac{1}{2}+\sum_{n=1}^{\infty}e^{-\pi n^2x}\right\}$$

is fundamentally an Abelian-Tauberian argument. In the present paper this argument is reproduced in a general set-up, although the author remarks that he has no specific instances to offer other than those well known. *A. L. Whiteman* (Los Angeles, Calif.).

Maass, Hans. Die Primzahlen in der Theorie der Siegelschen Modulfunktionen. Math. Ann. **124**, 87–122 (1951).

In the Hecke-Petersson theory of modular forms, a set of operators $\{T(m)\}$ is introduced each of which maps the family $\mathfrak{M}_k^{(1)}$ of modular forms of dimension $-k$ ($k\geqq 4$ and even) into itself. An end result is the theorem that it is possible to select a finite set of modular forms which constitutes a basis for $\mathfrak{M}_k^{(1)}$ and at the same time are eigenfunctions of $\{T_m\}$. Furthermore, the eigenvalues are Fourier coefficients of the basis functions and possess a multiplicative arithmetic property.

In the present paper the author makes an attempt to construct a similar theory for the Siegel modular forms of degree n. (For concepts and definitions, see Siegel, Math. Ann. **116**, 617–657 (1939); these Rev. **1**, 203; and Maass, ibid. **123**, 125–151 (1951); these Rev. **13**, 210.) First he defines a generalized operator $\tau(m)$:

$$f(Z)\,|\,\tau(m)=c_{nk}(m)\sum_{\sigma}f(Z)\,|\,\sigma,$$

where f is a modular form of dimension $-k$ and degree n, m is a positive integer, the $C_{nk}(m)$ are certain constants chosen so that in particular $\tau(m)$ reduces to $T(m)$ when $n=1$, σ runs over a set of representatives of the cosets of the modular group of degree n in the set of substitutions of order m, and

$$f(Z)\Big|\sigma=f(Z)\Big|\begin{pmatrix}A & B\\ C & D\end{pmatrix}=f(\sigma(Z))\,|\,CZ+D|^{-k}.$$

(σ is of order m if $\sigma'i\sigma=mi$, where

$$i=\begin{pmatrix}0 & E\\ -E & 0\end{pmatrix},$$

and the entries in A, B, C, D are integers.) The author restricts himself to the case $m=p$, prime, and must assume $k\geqq 2n+2$ in order to insure the convergence of the Poincaré series used in the proof.

The set $\mathfrak{M}_k^{(n)}$ of modular forms of dimension $-k$ and degree n is made into a metric space by means of the Petersson scalar product which Maass has generalized in the paper quoted above. Let $\mathfrak{S}_k^{(n)}$ denote the family of cusp forms in $\mathfrak{M}_k^{(n)}$; $\mathfrak{N}_k^{(n)}$, the orthogonal complement of $\mathfrak{S}_k^{(n)}$.

The Siegel operator Φ:

$$f(Z)|\Phi=\lim_{\lambda\to\infty} f\begin{pmatrix} Z^* & 0 \\ 0 & i\lambda \end{pmatrix},$$

maps $\mathfrak{M}_k^{(n)}$ onto $\mathfrak{M}_k^{(n-1)}$. Maass proves the existence of a unique decomposition $\mathfrak{M}_k^{(n)}=\mathfrak{S}_{k,0}^{(n)}+\cdots+\mathfrak{S}_{k,n-1}^{(n)}+\mathfrak{S}_k^{(n)}$, such that $\mathfrak{S}_{k,j}^{(n)}+\cdots+\mathfrak{S}_{k,j}^{(n)}$ is mapped one-to-one onto $\mathfrak{M}_k^{(j)}$ by Φ^{n-j}, $0\leqq j\leqq n-1$. Interest centers on $\mathfrak{S}_{k,0}^{(n)}+\mathfrak{S}_{k,1}^{(n)}$, the set mapped by Φ^{n-1} on $\mathfrak{M}_k^{(1)}$, the space of the usual modular forms. The author next establishes the commutativity $\Phi T(p)=T(p)\Phi$. This step requires a long, complicated proof.

According to the Petersson theory mentioned earlier [cf. Math. Ann. 116, 401–412 (1939)] there exist eigenfunctions $f_\nu|T(p)=a_\nu(p)f_\nu$, $\nu=1, 2, \cdots, \rho_1$. Define the forms g_ν by $g_\nu|\Phi^{n-1}=f_\nu$, $\nu=1, 2, \cdots, \rho_1$; these functions lie in $\mathfrak{S}_{k,0}^{(n)}+\mathfrak{S}_{k,1}^{(n)}$ and constitute a basis thereof. We have then

$$g_\nu|T(p)\Phi^{n-1}=[g_\nu|\Phi^{n-1}]|T(p)=f_\nu|T(p)$$
$$=a_\nu(p)f_\nu=a_\nu(p)g_\nu|\Phi^{n-1},$$

from which it follows that (*) $g_\nu|T(p)=a_\nu(p)g_\nu$, $1\leqq\nu\leqq\rho_1$, i.e., the g_ν are eigenfunctions of the set $\{T(p)\}$. The $\{a_\nu(p)\}$ are Fourier coefficients of f_ν; therefore, of g_ν. These three properties characterize the g_ν to within a multiplicative factor. By expanding both sides of (*) into Fourier series the author obtains a complicated equation $a_\nu(p)a_\nu(T)=\cdots$, which is intended to generalize the multiplication law $a_\nu(r)a_\nu(s)=\sum_{d|(r,s)}a_\nu(rs/d^2)\cdot d^{k-1}$ of the Hecke theory. For the case $n=2$, which is treated in detail in the last section of the paper and without the restriction $m=p$, the multiplication law is obtained in a simple form. *J. Lehner.*

Fogels, Ē. K. On an elementary proof of the prime number theorem. Latvijas PSR Zinātņu Akad. Fiz. Mat. Inst. Raksti. 2, 14–45 (1950). (Russian. Latvian summary)

The author proposes that an "elementary" proof of the prime number theorem should be carried out entirely within the domain of rational numbers. Thus he wishes also to remove from the work of A. Selberg and P. Erdös the presence of such "transcendental" elements as the functions $\log x$, $\log\log x$, x^σ, etc. This program is effected by replacing the use of e^x by

$$E(x)=1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^N}{N!}$$

where N is a suitable function of the total range X entering into the discussion. Then for $M=M(X)$ another such function of X (which serves as a sort of common denominator for the rational numbers in the proof) for any rational number $\xi\geqq 1$ let k be the integer such that $E(k/M)\leqq\xi<E((k+1)/M)$ and define $L(\xi)=k/M$. This function $L(x)$ is then used to replace $\log x$. Carrying out this replacement systematically, and following A. Selberg's method of proof very closely, the prime number theorem is derived in the form $\pi(x)\sim x/\sum_{n\leqq x}n^{-1}$. Proofs of a similar category are then derived for $\sum_{n\leqq x}\mu(n)=o(x)$, and $\sum_{n\leqq x}\mu(n)/n=o(1)$.

H. N. Shapiro (New York, N. Y.).

Čudakov, N. G., and Rodosskiĭ, K. A. New methods in the theory of Dirichlet's L-functions. Uspehi Matem. Nauk (N.S.) 4, no. 2(30), 22–56 (1949). (Russian)

This is an expository paper mainly devoted to the method of "density of zeros". This is the name attached by the Russian number-theorists to the application in prime-number theory of estimates of $N(\sigma, T)$, the number of zeros of an L-function in the rectangle $\sigma\leqq\Re(s)\leqq 1$, $-T\leqq\Im(s)\leqq T$, where $\sigma>\frac{1}{2}$ and $T\geqq 1$. The authors take these estimates of $N(\sigma, T)$ for granted and discuss only how they are applied. The paper consists of four sections.

In the first section the authors give a very complete proof of a result on trigonometric sums involving prime numbers which was announced earlier by Čudakov [Doklady Akad. Nauk SSSR (N.S.) 58, 1291–1294 (1947); these Rev. 9, 333]. Actually in this part of the paper the authors go much further in the direction of completeness than seems necessary, for they give detailed proofs of a number of preliminary theorems which are either immediate corollaries or obvious generalizations of well-known results. Thus Lemma 1 is a straightforward application of integration by parts, Lemma 2 is a direct corollary of the second mean-value theorem, Lemma 3 is a variant of the Fourier summation formula, and Theorem 1 is a rather routine generalization of a well-known theorem of van der Corput [Landau, Vorlesungen über Zahlentheorie, v. 1, Hirzel, Leipzig, 1927, Satz 352].

The second section is mainly devoted to Čudakov's proof of the Goldbach-Vinogradov theorem [Ann. of Math. (2) 48, 515–545 (1947); these Rev. 9, 11], starting from the relevant theorem on $N(\sigma, T)$ [op. cit., Theorem 2]. The authors sketch the proof of the essential approximation formula for $f(x)=\sum_{p>2}(\ln p)x^p$ [op. cit., Theorem 3] and the deduction therefrom of the asymptotic formula for the coefficients of $f^3(x)$. In addition they show in detail how the approximation formula for $f(x)$ leads to the theorem that almost all even natural numbers are sums of two odd primes. This latter proof is not found in Čudakov's paper and is attributed by the authors to V. Arhangel'skiĭ; it is similar to but slightly neater than the Hardy-Littlewood treatment [Proc. London Math. Soc. (2) 22, 46–56 (1923)]. The section concludes with a brief resume of a paper of Čudakov [Izvestiya Akad. Nauk SSSR. Ser. Mat. 12, 31–46 (1948); these Rev. 9, 499] on the finite differences of $\psi(x, k, l)$, where $\psi(x, k, l)$ has its usual meaning in prime-number theory.

In the third section the authors sketch Linnik's proof that if $k>1$ and $(l, k)=1$, then there exists a prime number p such that $p\equiv l \pmod k$ and $p<k^c$, where c is an absolute constant [Mat. Sbornik N.S. 15(57), 3–12, 139–178, 347–368 (1944); 16(58), 101–120 (1945); these Rev. 6, 260; 7, 146]. The reviewer regrets that the authors did not treat the material of this section with the completeness of detail which characterizes the first section, for even after this exposition Linnik's proof will remain an enigma to many number-theorists.

The last section has no direct connection with the method of density of zeros and was actually written by Linnik. In it he shows that the Lindelöf hypothesis is equivalent to asserting that as $T\to\infty$ the relation

$$\sum_{n=1}^{\infty}\tau(n)n^{iT}\exp\left\{-\tfrac{1}{4}\ln^2\frac{T}{2\pi en}\right\}=O(T^{1+\epsilon})$$

holds for any positive ϵ, where $\tau(n)$ denotes the number of divisors of n.

Although this paper is disappointing in some ways (mainly in the conciseness of its third section), it still is a very welcome addition to the literature of expository papers in analytic number theory. *P. T. Bateman.*

Barnes, E. S. The minimum of the product of two values of a quadratic form. II. Proc. London Math. Soc. (3) 1, 385–414 (1951).

Let $Q(x, y)=ax^2+bxy+cy^2$ be an indefinite quadratic form of positive determinant $D=b^2-4ac$. Excluding hereafter the trivial case where Q is a zero form, let M and $-N$ be the numerically least positive and negative values respectively of $Q(x, y)Q(u, v)$, where x, y, u, v are integers such that $xv-uy=1$. Setting $\lambda(Q)=D/M$ and $\mu(Q)=D/N$, the existence of integers x, y, u, v with $xv-uy=1$ for which $-D/m\leqq Q(x, y)Q(u, v)\leqq D/l$ is equivalent to the condition that either $\lambda(Q)\geqq l$ or $\mu(Q)\geqq m$.

In a previous paper [same Proc. (3) 1, 257–283 (1951); these Rev. 13, 627] the author obtained a sequence of forms $f_{-1}, f_0, f_1, f_3, f_5, \cdots$ for which $\lambda_n=\lambda(f_n)$ strictly increases to $\lambda_\infty=(13\sqrt{10}+3)/6$, such that $\lambda(Q)\geqq\lambda_\infty$ or $\mu(Q)\geqq\lambda_\infty$ unless Q is equivalent to a multiple of some f_n. In this paper asymmetric bounds are obtained for $\lambda(Q)$ and $\mu(Q)$. A second sequence of forms $\psi_0, \psi_2, \psi_4, \cdots, \psi_\infty$ is found for which $\mu_n=\mu(\psi_n)$ strictly increases to $\mu_\infty=\mu(\psi_\infty)=2(\sqrt{10}+1)$, as well as a form f_∞ for which $\lambda(f_\infty)=\lambda_\infty$, and it is shown that $\lambda(Q)>\lambda_\infty$ or $\mu(Q)>\mu_\infty$ unless Q is equivalent to a multiple of one of the f_n or ψ_n (including f_∞ and ψ_∞). To prove this result, the author first shows that the condition "$\lambda(Q)>l$ or $\mu(Q)>m$" is equivalent (with one minor restriction) to the falsehood of at least one of an infinite set of inequalities which involve certain continued fractions associated with the sequence of reduced forms equivalent to Q. The major difficulty of the proof consists of showing that if all of these inequalities hold with $l=\lambda_\infty$ and $m=\mu_\infty$, then these continued fractions must have a very specific structure, and further that this structure can arise only from one of the f_n or ψ_n (including f_∞ and ψ_∞). The author also proves that the constants λ_∞ and μ_∞ are in a certain sense the best possible. All of the proofs in this paper are independent of the results of the previous paper, and so furnish another proof of those results. *I. Reiner* (Urbana, Ill.).

Barnes, E. S. The minimum of the product of two values of a quadratic form. III. Proc. London Math. Soc. (3) 1, 415–434 (1951).

We shall use the notation of the preceding review. In this paper the author shows that $\lambda(Q)\geqq 4$ and $\mu(Q)\geqq 5$ for all Q. In particular, he proves that $\lambda(Q)\geqq\lambda_\infty=(100+13\sqrt{13})/27$ unless Q is equivalent to a multiple of some one of a sequence of forms whose λ's increase from 4 to λ_∞. It is also shown that $\mu(Q)\geqq\mu_\infty=2(\sqrt{5}+1)$ unless Q is equivalent to a multiple of some one of another sequence of forms whose μ's increase from 5 to μ_∞. Furthermore, the author proves his results to be best possible. The same approach is used here as in the preceding paper; however, these results are somewhat more difficult to prove than the previous ones.

I. Reiner (Urbana, Ill.).

Gastinger, Walter. Über die untere Grenze der positiven Werte reeller quadratischer Formen. Monatsh. Math. 56, 49–60 (1952).

The author is concerned with finding numbers such that the inequality $0<P(x_1, \cdots, x_n)\leqq p_n|D|^{1/n}$ is soluble in integers $x_1, \cdots, x_n$ for every indefinite quadratic form P with real coefficients and non-zero determinant D. The best possible values of p_n are known when n is 2 or 3, and Blaney [J. London Math. Soc. 23, 153–160 (1948); these Rev. 10, 511] proved in a lemma that 2^{n-1} is always an admissible value for p_n. The author follows Blaney's method, and by a more detailed treatment of the inequalities he obtains smaller values for p_n for individual values of n, and the estimate $(4/3)^{(n-1)/2}$ for large n. He also points out that the example $P=x_1x_n+x_2x_{n-1}+\cdots$ shows that p_n cannot be taken less than 2 for even n, or less than $2^{1-1/n}$ for odd n.

H. Davenport (London).

Kanagasabapathy, P. Note on Diophantine approximation. Proc. Cambridge Philos. Soc. 48, 365–366 (1952).

A well-known theorem of Minkowski states: If θ is any irrational number and if α is a real number, not of the form $x-\theta y$ (x and y integers), then there are infinitely many integers x, y, with $y\neq 0$, such that $|(x-\theta y-\alpha)y|<\frac{1}{4}$. Moreover, it is known that the constant $\frac{1}{4}$ on the right is the best possible [cf. Koksma, Diophantische Approximationen, Springer, Berlin, 1936, p. 77]. The author shows that the constant $\frac{1}{4}$ still is the best possible if only one solution (x, y) of the Diophantine inequality is demanded instead of an infinity. *J. Popken* (Utrecht).

Koksma, J. F. Sur l'approximation des nombres irrationnels sous une condition supplémentaire. Simon Stevin 28, 199–202 (1951).

Let θ be irrational, and let $s\geqq 1$, a and b be integers. It is shown that to each $\epsilon>0$ there correspond infinitely many pairs of integers p, q with $q>0$ such that

$$|\theta-p/q|<s^2(1+\epsilon)/\sqrt{5}q^2 \text{ and } p\equiv a \pmod s,\ q\equiv b \pmod s.$$

This theorem, in which the constant $\sqrt{5}$ is clearly best possible, is an improvement on the corresponding theorem of S. Hartman [Colloquium Math. 2, 48–51 (1949); these Rev. 12, 807] in which the above inequality is replaced by $|\theta-p/q|<2s^2/q^2$. In the case that not both of a and b are divisible by s, it is also shown that there are infinitely many pairs p, q for which $|\theta-p/q|<s^2/4|q(q-b)|$ and $p\equiv a \pmod s$, $q\equiv b \pmod s$. *W. J. LeVeque.*

Descombes, Roger, et Poitou, Georges. Sur certains problèmes d'approximation. C. R. Acad. Sci. Paris 234, 581–583 (1952).

Soit ξ un nombre réel positif, s un nombre naturel et soient a et b des nombres entiers. A la suite de S. Hartman [Colloquium Math. 2, 48–51 (1949); ces Rev. 12, 807] les auteurs à l'aide de la théorie des fractions continues, étudient la limite inférieure $K(\xi, s, a, b)$ de $|v(\xi v-u)|$ pour les fractions u/v à termes positifs congrus à a et b respectivement modulo s. Le référent, en utilisant une méthode tout à fait différente, a démontré $K(\xi, s, a, b)\leqq 5^{-1/2}s^2$ (et bien $\leqq\frac{1}{4}s^2$, si l'on admet des valeurs négatives pour u et v) [voir l'analyse ci-dessus]. Des résultats classiques de Hurwitz, il suit facilement qu'on ne peut pas améliorer ce résultat (il suffit de poser $a=b=0$ ou de prendre $s=1$). En supposant que les trois nombres a et b et s sont premiers entre eux et (si le référent a bien compris) en supposant aussi que $s\geqq 2$, les auteurs montrent que toujours $K(\xi, s, a, b)\leqq 4s^2/9$, pendant que d'autres résultats sont annoncés. Ils remarquent qu'on peut se limiter au cas $a=1$, $b=0$ quelque soit le nombre $s\geqq 2$.

J. F. Koksma (Amsterdam).

Davenport, H., and Erdös, P. Note on normal decimals. Canadian J. Math. 4, 58–63 (1952).

A number $0.a_1a_2\cdots$ expressed as a decimal is said to be normal (Borel), if every fixed combination among the 10^k possible combinations of k digits occurs with the frequency $1/10^k$. Let $N(t)$ denote the number of times the said combination occurs among the first t digits; then the condition

is that $\lim_{t\to\infty} N(t)/t = 1/10^k$. A positive integer $g = a_1a_2\cdots a_l$ expressed in the decimal scale is said to be (ϵ, k) normal [Besicoyitch, Math. Z. 39, 146–156 (1934)], if any combination of $h \leqq k$ digits occurs in the sequence $a_1\cdots a_l$ a number of times which lies between $(1-\epsilon)10^{-h}l$ and $(1+\epsilon)10^{-h}l$.

The authors prove the following theorems 1 and 2 which contain known results of Champernowne [J. London Math. Soc. 8, 254–260 (1933)] and Besicovitch [loc. cit.] and the first of which was conjectured by Copeland and Erdös [Bull. Amer. Math. Soc. 52, 857–860 (1946); these Rev. 8, 194]. Theorem 1. Let $f(x)$ be any polynomial in x, all of whose values for $x = 1, 2, \cdots$ are positive integers. Then the decimal $0.f(1)f(2)f(3)\cdots$, where $f(n)$ is written in the scale of 10 is normal. Theorem 2. For any $\epsilon > 0$ and $k \geqq 1$ (k an integer) almost all the numbers $f(1), f(2), \cdots$ are (ϵ, k) normal, i.e. the number of indices $n \leqq x$ for which $f(n)$ is not (ϵ, k) normal is $o(x)$ as $x \to \infty$ for fixed ϵ and k. The proof follows the method with which Weyl's criterion for uniform distribution is proved [Math. Ann. 77, 313–352 (1916)].

J. F. Koksma (Amsterdam).

Maxfield, John E. A short proof of Pillai's theorem on normal numbers. Pacific J. Math. 2, 23–24 (1952).

The author gives a simple proof of the following theorem: The necessary and sufficient condition that a number α be normal to the base r is that it be simply normal to the bases $r, r^2, \cdots$.

P. Erdös (Los Angeles, Calif.).

ANALYSIS

Mohr, Ernst, und Noll, Walter. Eine Bemerkung zur Schwarzschen Ungleichheit. Math. Nachr. 7, 55–59 (1952).

The authors obtain a generalization of the inequality $(\int_a^b f)^2 \leqq (b-a)\int_a^b f^2$ for a function $f(t)$ with a continuous nth derivative. This is a consequence of the following identity:

$$(*)\quad \left(\int_a^b f\,dt\right)^2 = (b-a)\sum_{\nu=0}^{n-1}\frac{(-1)^\nu}{\nu!(\nu+1)!} \times \int_a^b |f^{(\nu)}(t)|^2[(b-t)(t-a)]^\nu dt + (-1)^n R_n,$$

$$R_n = \int\cdots\int\left|\int_{x_1}^{y_1} f^{(n)}(t)\,dt\right|^2 dx_1dy_1\cdots dx_ndy_n,$$

where the integration in R_n is over

$$a \leqq x_n \leqq \cdots \leqq x_1 \leqq y_1 \leqq \cdots \leqq y_n \leqq b,$$

so that R_n can also be written in the form

$$2(n!)^{-2}\int_{a\leqq t\leqq s\leqq b}\int f^{(n)}(s)f^{(n)}(t)[(b-s)(t-a)]^n ds\,dt.$$

The proof is by induction on n. Since $R_n \geqq 0$, $(\int f)^2$ is at most or at least equal to the sum in (*) according as n is even or odd. The authors also show that $R_n \to 0$ if the power series of $f(z)$ has a radius of convergence at least equal to $\frac{1}{2}(b-a)$ at every point of $[a, b]$.

R. P. Boas, Jr.

Tagamlitzki, Y. Über die Abelsche Interpolationsreihe. Annuaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1. 46, 385–443 (1950). (Bulgarian. German summary)

The results of this paper were summarized in Doklady Akad. Nauk SSSR (N.S.) 80, 17–20 (1951); these Rev. 13, 329.

R. P. Boas, Jr. (Evanston, Ill.).

Viola, Tullio. Sur l'approximation des fonctions continues. J. Math. Pures Appl. (9) 31, 91–101 (1952).

The author establishes some properties of Ghizzetti's method of interpolation [Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 80, 225–230 (1945); these Rev. 8, 458]. In this method one has an orthonormal set $\{\phi_k(x)\}$, complete in $L^2(a, b)$ and puts $\Phi_k(x) = \int_a^x \phi_k(t)\,dt$; if $n+1$ points in the (x, y)-plane are given, $a = x_0 < x_1 < \cdots < x_n = b$, and one considers linear combinations $y_0 + \sum_{k=0}^{n+p-1} c_k\Phi_k(x) = g_p(x)$, there is a $g_p(x)$ minimizing $\int_a^b |g_p'(x)|^2\,dx$ among all such functions whose graphs pass through all (x_k, y_k); Ghizzetti showed that as $p \to \infty$, the limit of $g_p(x)$ is the function $\gamma(x)$ whose graph is the polygon through the given points, and that $g_p'(x) \to \gamma'(x)$ in mean. Viola gives sufficient conditions for $g_p'(x)$ to approach $\gamma'(x)$ at a given point or in an interval. Under more restrictive hypotheses he shows that if a given continuous $f(x)$ is approximated by a polygon and then the polygon is approximated by Ghizzetti's method by a $g_p(x)$, it will also be true that $g_p'(x)$ approximates $f'(x)$.

R. P. Boas, Jr. (Evanston, Ill.).

Olevskiĭ, M. N. On approximation of a continuous function on a given interval by piecewise linear functions. Doklady Akad. Nauk SSSR (N.S.) 82, 193–196 (1952). (Russian)

Soit $f(x)$ une fonction continue sur $[a, b]$, et soit $\varphi_n(x)$ une fonction représentant une ligne polygonale lorsque $x \in [a, b]$, avec $n+1$ sommets (l'abcisse de P_0 étant a, celle de P_n étant b). Posons $\Delta[f, \varphi_n] = \max_{a\leqq x\leqq b}|f(x) - \varphi_n(x)|$. Désignons par S_n la famille des courbes φ_m (représentées par $\varphi_m(x)$) pour lesquelles $m \leqq n$. La courbe $L_n \in S_n$ telle que $\Delta[f, L_n] = \inf[f, \varphi_m]$ $(\varphi_m \in S)$ est dite la meilleure selon Tchebycheff, de la famille S_n pour f. Voici quelques problèmes posés et résolus par l'auteur: De toutes les courbes de S_n inscrites dans la courbe AB (représentée par $f(x)$) trouver la meilleure selon Tchebycheff pour f; de toutes les courbes de S_n trouver la meilleure selon Tchebycheff pour f; de toutes les courbes de S_n passant par A trouver la meilleure selon Tchebycheff, etc.

S. Mandelbrojt.

Redheffer, R. M. Moments which are integers. Proc. Amer. Math. Soc. 3, 282 (1952).

A theorem due to Pólya and Carlson [see Bieberbach, Lehrbuch der Funktionentheorie, vol. II, Teubner, Leipzig-Berlin, 1927, p. 327] asserts that, if $f(z) = \sum C_n z^n$ is regular in a simply connected region about the origin whose mapping radius is larger than one, and the C_n are integers, then $f(z)$ is rational. This has often been used to discuss the possibility that a sequence $\{C_n\}$ obtained in a special way could be integral valued. The author studies this for the sequence $C_n = \int_0^a t^n d\alpha(t)$ where $\alpha(t)$ is of bounded variation with infinitely many points of increase on $[0, a]$, obtaining the result that C_n cannot be integral for each N unless $a \geqq 4$. An example shows that $a = 4$ is possible.

R. C. Buck.

Ahiezer, N. I. A. A. Markov's problem of moments relative to an arbitrary number of intervals. Ukrain. Mat. Žurnal 1, no. 3, 41–50 (1949). (Russian)

The author solves the moment problem $c_k = \int_E u^k f(u)du$, $k = 0, 1, \cdots, n-1$, $|f(u)| \leqq L$, where E consists of a finite number m of nonoverlapping intervals in $(-1, 1)$. Necessary

and sufficient conditions for a solution consist of the non-negativeness of 2^n quadratic forms. *R. P. Boas, Jr.*

Georgiev, G. Mechanical quadrature formulas for multiple integrals with a minimal number of terms. Doklady Akad. Nauk SSSR (N.S.) **83**, 521–524 (1952). (Russian)

The problem is to find, for a given plane region R and the integer n, a minimal number N of points (x_i, y_i) and coefficients λ_i such that $\iint_R \phi(x,y)dxdy = \sum_{i=1}^{N} \lambda_i \phi(x_i, y_i)$ for each polynomial ϕ of degree $\leqq n$. The author shows that $N=3$ for $n=2$ and that $N=4$ for $n=3$ and an R with a center of symmetry and indicates the corresponding points (x_i, y_i). *G. G. Lorentz* (Kingston, Ont.).

***Panov, D. Yu. On a formula of mechanical quadratures.** Trudy Mat. Inst. Steklov., v. 38, pp. 317–320. Izdat. Akad. Nauk SSSR, Moscow, 1951. (Russian) 20 rubles.

The author presents a formula of mechanical quadratures using three ordinates, not necessarily equally spaced, reducing to Simpson's formula in the case of equal spacing. If the spacing is nearly equal, so that $(x_2-x_1)/(x_1-x_0)=1+\epsilon$ where ϵ is small and higher powers of ϵ are neglected, the formula becomes

$$\int_{x_0}^{x_2} y\,dx = \frac{h}{3}\left[y_0\left(1-\frac{\epsilon}{2}\right)+xy_1\left(1+\frac{\epsilon}{2}\right)+y_2\left(1+\frac{3\epsilon}{2}\right)\right].$$

He shows how this formula can be used in the solution of partial differential equations with two families of characteristic curves in the xy-plane. *W. E. Milne.*

Koksma, J. F. A Diophantine property of some summable functions. J. Indian Math. Soc. (N.S.) **15** (1951), 87–96 (1952).

Let $g(t)$ be a periodic function in L^2 of period 1. Let $\sum_{k=-\infty}^{\infty} c_k e^{2\pi i k t}$ be the Fourier expansion of $g(t)$. Assume that

$$(1) \qquad \sum_{k=1}^{\infty}\left(|c_k|^2 \sum_{d|k} \frac{\log^2 d}{d}\right) < \infty.$$

Then the author proves that for almost all x

$$(2) \qquad \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} g(nx) = \int_0^1 g(t)\,dt.$$

The author remarks that (1) can be replaced by

$$\sum_{k=1}^{\infty} |c_k|^2 (\log\log k)^2 < \infty.$$

The interesting question whether (2) holds for every periodic $g(t)$ in L^2 is left open; in fact, the reviewer did not succeed in finding a periodic $g(t)$ in L which does not satisfy (2). *P. Erdős* (Los Angeles, Calif.).

Calculus

Scorza Dragoni, Giuseppe. Un'osservazione sulla derivata di una funzione composta. Rend. Sem. Mat. Univ. Padova **20**, 462–467 (1951).

The observations center about the following result, a precise statement and proof of which are given: if $z(x, y)$ is measurable with respect to x and continuous with respect to y and if the four derivatives involved exist almost everywhere, then $z'[x, \rho(x)] = z_x[x, \rho(x)] + z_y[x, \rho(x)]\rho'(x)$. *J. M. Thomas* (Paris).

Lauffer, R. Eine Hermitesche Rekursionsformel. Jber. Deutsch. Math. Verein. **55**, Abt. 1, 68–69 (1952).

Let

$$D^r_{s,t} = \frac{\partial^{r+s}}{\partial x^s \partial y^t}(xy-1)^{r+s+t}.$$

In this paper it is proved that

$$D^r_{s,t} = s!t!\frac{(xy-1)^r}{(r+s)!(r+t)!}D^{\rightarrow}_{r+s,\,r+t},$$

and that

$$D^r_{s,t} = s!t!\frac{(xy-1)^r x^{t-s} y^{s-t}}{(r+s)!(r+t)!}D^{\rightarrow}_{r+t,\,r+s}.$$

If one sets $r=p+q$, $s=n-p$, $t=m-q$, the two formulas reduce respectively to a formula given by Hermite and a variation of it given by Koschmieder [same Jber. **54**, 52–54 (1950); these Rev. **12**, 164]. *A. E. Taylor.*

Phipps, C. G. Maxima and minima under restraint. Amer. Math. Monthly **59**, 230–235 (1952).

Mitrović, Dragiša. Sur une égalité d'intégrales. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **6**, 193–200 (1951). (Serbo-Croatian. French summary)

Let $f(z)$ be meromorphic in the semicircle $|z|\leqq R$, $\Im z \geqq 0$, with zeros a_ν of order α_ν, $(\alpha_1+\cdots+\alpha_n=N)$ and poles b_ν of order β_ν, $(\beta_1+\cdots+\beta_p=P)$. Let $h(z)$ be holomorphic in the closed semicircle. It is proved that

$$\int_{-R}^{R} h'(x)\log f(x)dx + iR\int_0^{\pi} e^{i\theta}h'(Re^{i\theta})\log f(Re^{i\theta})d\theta$$
$$= 2\pi i\left((N-P)h(R) - \sum_1^n \alpha_\nu h(a_\nu) + \sum_1^p \beta_\nu h(b_\nu)\right).$$

This equality is split into real and imaginary parts to give the author's principal theorem. *A. W. Goodman.*

Guglielmino, Francesco. Calcolo di integrali singolari mediante il teorema dei residui. Matematiche, Catania **6**, 97–112 (1951).

This paper is concerned with the evaluation of certain infinite integrals by means of contour integration over the complex plane cut along $(0, \infty)$, using in turn the integrands

$$\frac{P(z^4)}{Q(z^4)}z^s e^{(i-1)z}, \quad \frac{P(z^2)}{Q(z^2)}z^s e^{iz}, \quad \frac{P(z)}{Q(z)}z^s \log z,$$

taken over appropriate contours. Here $P(z)$, $Q(z)$ are given polynomials, the degree of $Q(z)$ being the greater, and usually $s>-1$ but occasionally $s=-1$ and is frequently bounded above. The range of the real integrals evaluated is $(0, \infty)$ and their integrands are combinations of the integrands of standard integrals to be found in text-books on the complex variable. *R. Wilson* (Swansea).

Toscano, Letterio. Sulla decomposizione in fattori simbolici della potenza dell'operazione di derivazione. Anais Fac. Ci. Porto **35**, 5–13 (1951).

The author establishes the symbolic relation

$$x^{m(n-1)}\left(\frac{d}{dx}\right)^{mn} = m^{mn}\left[x^{mn-1}\left(\frac{d^n}{dx^m}\right)^m\right]$$

and in the cases $m=2$, 3 also the corresponding formulas for $(d/dx)^{2n+1}$, $(d/dx)^{2n+1}$, $(d/dx)^{2n+2}$. He shows that the case $m=2$ may serve to establish the relationship between

(generalised) Laguerre and Hermite polynomials, and $m=3$ between Laguerre polynomials and certain polynomials investigated by Humbert. *A. Erdélyi* (Pasadena, Calif.).

Hines, Jerome. Foundations of operator mathematics. Math. Mag. **25**, 251–261 (1952).

***Kočin, N. E. Vektornoe isčislenie i načala tenzornogo isčisleniya. [Vector calculus and the elements of tensor calculus.]** 7th ed. Izdat. Akad. Nauk SSSR, Moscow, 1951. 426 pp. $2.50.

This edition differs from the sixth [GTTI, Moscow-Leningrad, 1938] only in a change of notation and correction of known errors and misprints. There have been no essential changes since the 4th edition in 1934.

Theory of Sets, Theory of Functions of Real Variables

Sierpiński, Waclaw. Dernières recherches et problèmes de la théorie des ensembles. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **10**, 1–11 (1951).

Sierpiński, Waclaw. Sur quélques propriétés des familles d'ensembles. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. **42** (1949), 30–35 (1952). (French. Polish summary)

A study is made of conditions under which a family F of sets each of which is the sum of two disjoint sets of F has the property that each set of F is the sum of an infinity of disjoint sets of F—also of the related questions in which the word "disjoint" is replaced, respectively, by "distinct" and by "distinct from itself." For example, it is shown that if F is an arbitrary family of countable sets each of which is the sum of two disjoint sets of F, then each set of F is the sum of an infinity of disjoint sets of F. However, this conclusion does not follow without the assumption of countability on the sets of F, even though F itself is a countable family.

G. T. Whyburn (Charlottesville, Va.).

Sierpiński, W. Sur un problème de M. J. Novák. Czechoslovak Math. J. **1(76)**, 97–101 (1951) = Čehoslovack. Mat. Ž. **1(76)**, 117–122 (1951).

All sets concerned are to be sets of natural numbers. Of two sets A and B the notation $B\supset^* A$ is to mean that $A-B$ is finite and $B>^* A$ to mean that $A-B$ is finite and $B-A$ infinite. Consider the following two problems, N and L, problem N having been proposed by J. Novák recently and problem L by N. Lusin in 1947. Problem N: Can a transfinite sequence of infinite sets N_ξ be found for $\xi<\Omega$ such that $N_\xi\supset^* N_\eta$ for all $\xi<\eta<\Omega$ and yet for no infinite set A does $N_\xi\supset^* A$ for all $\xi<\Omega$? Problem L: Can a transfinite sequence of infinite sets N_ξ be found for $\xi<\Omega$ such that $N_\xi>^* N_\eta$ for all $\xi<\eta<\Omega$ and yet for no infinite set A does $N_\xi>^* A$ for all $\xi<\Omega$? The author proves these two problems equivalent. He has previously shown [Fund. Math. **35**, 141–150 (1948); these Rev. **10**, 689] that problem L can be answered affirmatively under the continuum hypothesis. Thus problem N can also; this is reproved by direct construction. *W. Gustin* (Bloomington, Ind.).

Sierpiński, Waclaw. Sur les fonctions continues d'une variable ordinale. Fund. Math. **38**, 204–208 (1951).

Let φ be an ordinal number without an immediate predecessor. The author defines that the sequence of ordinals $\{a_\xi\}$, $\xi<\varphi$, has the ordinal λ as its limit, if for every $\nu<\lambda$ there exists a $\mu<\varphi$ so that, for $\mu<\xi<\varphi$, $\nu<a_\xi\leqq\lambda$. A function $f(\xi)$ is said to be continuous if $\lim_{\xi<\varphi} f(\xi)=f(\varphi)$. The author then proves various theorems on continuous functions. Let α be any countable ordinal number, $f(\xi)$ any function defined for the ordinals $\xi<\alpha$. Then $f(\xi)=\lim f_n(\xi)$, $n=1, 2, \cdots$, where the $f_n(\xi)$ are continuous functions. On the other hand the function $f(\xi)=\xi+1$, $\xi<\Omega$ is not the limit of any transfinite sequence of continuous functions. *P. Erdős.*

Sierpiński, Waclaw. Sur quelques résultats nouveaux concernant l'hypothèse du continu. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **10**, 406–411 (1951).

The author proves, without using transfinite numbers, that the continuum hypothesis is equivalent to the following statement: Let S be the euclidean space of three dimensions. One can decompose S into the union of three sets S_i, $i=1, 2, 3$ so that for every i the intersection of S_i with every line parallel to the axis OX_i is finite. The author remarks that if $c=\aleph_1$ then the plane is the sum of countably many curves (a curve is a set of points which is intersected in at most one point by every line parallel to a given one). The author cannot decide that $c=\aleph_1$ is a consequence of the statement that the plane is the sum of countably many curves. The author states the following unsolved problem: Do there exist three lines D_i, $i=1, 2, 3$, in the plane so that the plane is the sum of three sets E_i, where every line parallel to D_i intersects E_i in a finite set. *P. Erdős.*

Ohkuma, Tadashi. A note on the ordinal power and the lexicographic product of partially ordered sets. Kōdai Math. Sem. Rep. **1952**, 19–22 (1952).

The author gives a new definition of ordinal power for partially ordered sets. The usual order [see G. Birkhoff, Lattice theory, 2nd ed., New York, 1948; these Rev. **10**, 673] is used in a subset of the space of all functions from X into Y. An element y_0 of Y is chosen; a function f is in ${}^X Y\langle y_0\rangle$ if $\{x\,|\,f(x)\neq y_0\}$ satisfies the descending chain condition. For each y_0 the resulting set is partially ordered; it is a chain if X and Y are chains. ${}^X Y\langle y_0\rangle$ is isomorphic to ${}^X Y\langle y_1\rangle$ if there is an automorphism φ of Y such that $\varphi(y_0)=y_1$. If the automorphism group of Y is transitive, Y is called homogeneous; then ${}^X Y\langle y_0\rangle$ is independent of y_0. The definition and results are extended to lexicographic products.

M. M. Day (Urbana, Ill.).

Ohkuma, Tadashi. On discrete homogeneous chains. Kōdai Math. Sem. Rep. **1952**, 23–30 (1952).

Continuing the note reviewed above, this note considers only chains; that is, simply ordered sets, which are homogeneous in the above sense. Such a chain is called discrete if each element has a unique smallest successor (and therefore a unique largest predecessor), and is called absolutely discrete if it has no non-trivial homogeneous non-discrete subchains. Let J be the chain of all integers; then an ordinal product $M\circ J$ is a discrete homogeneous chain if and only if M is a homogeneous chain. J is absolutely discrete; also if $X=Y\circ Z$, then X is absolutely discrete if and only if both Y and Z are (or if one is trivial and the other is X). If K' is the dual of an ordinal K and if $H_K={}^{K'}J\langle 0\rangle$, then H_K is absolutely discrete; it is shown that every absolutely discrete homogeneous chain is isomorphic to some H_K, and that every discrete homogeneous chain can be decomposed into an ordinal product $M\circ H_K$, where M is non-discrete.

M. M. Day (Urbana, Ill.).

Neves Real, Luis. The notion of "filter" and its relations with the theory of limits and the definition of real numbers. Gaz. Mat., Lisboa **12**, no. 50, 39–46 (1951). (Portuguese)

McShane, E. J. Partial orderings and Moore-Smith limits. Amer. Math. Monthly **59**, 1–11 (1952).

This paper is a careful, readable exposition, with many applications, of Moore-Smith convergence. The author defines directed sets and nets (functions on directed sets) and by use of these notions gives precise and simple definitions of limits of functions, sequences and double sequences, of length of arc, and of the Riemann integral (both refinement and mesh definitions). The definition of subnet is credited to the reviewer [Duke Math. J. **17**, 277–283 (1950); these Rev. **12**, 194]. Since publication it has been pointed out to the author (and to the reviewer) that the definition occurs in E. H. Moore [General analysis, vol. I, pt. II, Amer. Philos. Soc., Philadelphia, 1939, ch. I]. Using this definition the hoped-for characterizations of compactness are proved. The paper ends with preliminary discussion of a form of convergence in ordered spaces. *J. L. Kelley.*

Pauc, Christian. Mesure et prétopologie. Les théorèmes forts de Vitali établis sous des conditions de limitation locale de la dilatation en halo. C. R. Acad. Sci. Paris **234**, 1119–1120 (1952).

Pauc, Christian. Adaptation d'une mesure à une prétopologie. Passage de la Propriété Forte de Vitali réduite à la propriété complète. C. R. Acad. Sci. Paris **234**, 1242–1243 (1952).

There is assumed a basic set R, a σ-algebra $\mathfrak{M}$ of measurable subsets M of R, a measure function μ on $\mathfrak{M}$, relative to which $\mathfrak{M}$ is complete, and the class $\mathfrak{N}$ of sets N of zero measure which form a σ-ideal. Relative to points x of a subset X, there is defined the notion of a Moore-Smith sequence of sets M_ι converging to x, the only requirement of this notion being that if the sequence M_ι converges to x so does every confinal sequence. The generalized D(enjoy) interior of a set X: $I(X)$ consists of those points of X such that the sets M_ι of any sequence converging to x ultimately belong to X; the generalized D(enjoy) closure of X: $F(X)$ consists of the points y for which there exists a sequence M_ι converging to y, all of whose terms intersect X. For any given set E, there are determined the sets G such that $E \cdot G \subseteq I(G)$ and the sets A such that $E \cdot A \supseteq F(A)$. The sets G determine an external topology for E. $\mathfrak{B}$-fin (Vitali) coverings are defined in terms to certain sets V, X being covered if for every x of X there exists a Moore-Smith sequence V_ι converging to x. In this note the sets V consist of sets A. By using the function Δ (function of disentanglement) of A. P. Morse and the notion of halo (the halo $H(V_0)$ is the union of the V intersecting V_0 for which $\Delta(V) \leqq \alpha\Delta(V_0)$ for some fixed $\alpha \geqq 1$) it is possible to state a Vitali restricted theorem (reduction of a $\mathfrak{B}$-fin covering of any subset X of E to a denumerable disjoint covering modulo $\mathfrak{B}$), and a strong Vitali theorem (for every $\mathfrak{B}$-fin covering of any subset X of E and $\epsilon > 0$ there exists a denumerable disjoint covering of X modulo $\mathfrak{N}$ up to an excess of ϵ). To obtain theorems on the derivatives of indefinite integrals the notion of Borel measurable sets is generalized. If C stands for the trace of a set G on E, then these sets are the measurable subsets Z' of E such that for every $\epsilon > 0$, there exists a C for which $\mu(Z' - C) < \epsilon$. The measure μ is then "adapted" to the sets $M' = M(R - E) + Z'$. The sets Z' turn out to be the measurable subsets of E for which the strong Vitali theorem is valid. Every μ-integral admits except for a set N a $\mathfrak{B}$-derivative equal to a μ'-integrand. The Denjoy theory [Amer. J. Math. **73**, 314–356 (1951); these Rev. **12**, 685] is subsumed under this theory by proper specialization.

T. H. Hildebrandt.

Choquet, Gustave. Propriétés fonctionnelles des capacités alternées ou monotones. Exemples. C. R. Acad. Sci. Paris **234**, 498–500 (1952).

Cette note est la troisième d'une série consacrée aux capacités [mêmes C. R. **234**, 35–37, 383–385 (1952); ces Rev. **13**, 555, 633]. Dans la première partie I l'auteur indique des opérations sur ensembles, transformant une capacité en une capacité du même ordre. II consiste en exemples de capacités d'ordre $\mathfrak{A}_\infty$ et $\mathfrak{M}_\infty$. III caractérise les mesures de Radon positives parmi les capacités. La souplesse et la généralité du nouvel instrument apparaissent clairement sur les résultats formulés avec concision et sans démonstrations. Nous ne reproduisons ici que quelques spécimens, laissant entrevoir les applications à la théorie du potentiel, à la géométrie, au calcul des probabilités, à la théorie de la mesure. I—E et F désignent deux espaces topologiques, $\mathfrak{E}$ et $\mathfrak{F}$ deux ensembles (familles) de parties (sous-ensembles) de E et F respectivement. (i) Une application $Y = \varphi(X)$ de $\mathfrak{E}$ dans $\mathfrak{F}$ est un $\bigcup$-homomorphisme continu à droite si $\mathfrak{E}$ est additif et si : a) Pour tout $A \in \mathfrak{E}$ et tout voisinage V de $B = \varphi(A)$, il existe un voisinage U de A tel que, pour tout $X \in \mathfrak{E}$ vérifiant $X \subset U$, on ait $\varphi(X) \subset V$ (Propriété de semi-continuité supérieure pour correspondances); b) $\varphi(A' \cup A'') = \varphi(A') \cup \varphi(A'')$ quand A' et $A'' \in \mathfrak{E}$. Proposition : Si f est une capacité d'ordre $\mathfrak{A}_\infty$ sur $\mathfrak{F}$, la fonction $e(X) = f(Y)$ est une capacité d'ordre $\mathfrak{A}_\infty$ sur $\mathfrak{E}$. Exemple important : E est le produit de F et G séparés et φ est la projection canonique de E sur F. Tout élément de $\mathfrak{E}$ est compact, $\mathfrak{F} = \varphi(\mathfrak{E})$, pour tout compact K de E, $(\varphi(K) \in \mathfrak{F})$ implique $K \in \mathfrak{E}$. (ii) Une application $Y = \varphi(X)$ de $\mathfrak{E}$ dans $\mathfrak{F}$ est un $\bigcap$-homomorphisme continu à droite si $\mathfrak{E}$ est multiplicatif, si a) ci-dessus reste vrai et si b) $\varphi(A' \cap A'') = \varphi(A') \cap \varphi(A'')$ quand A' et $A'' \in \mathfrak{E}$. Proposition : Si f est une capacité d'ordre $\mathfrak{M}_\infty$ sur $\mathfrak{F}$, la fonction $e(X) = f(Y)$ est une capacité d'ordre $\mathfrak{M}_\infty$ sur $\mathfrak{E}$. Exemple : F est l'espace topologique des compacts de E et pour tout $X \in \mathfrak{E}$, $\varphi(x) =$ ensemble des compacts de X. II—$\mathfrak{E} =$ ensemble $\mathfrak{K}(E)$ des compacts K de l'espace E: (i) Exemples de capacités d'ordre $\mathfrak{A}_\infty$: (1) La capacité classique pour la fonction de Green d'un domaine E de R^n, (2) E étant le plan, $f(K)$ désigne la mesure (invariante classique) de l'ensemble des droites du plan qui rencontrent K, (3) E est un ensemble fini de parties de pile ou face, pour tout $K \subset E$, $f(K) =$ probabilité d'avoir pile une fois au moins sur K, (4) μ est une mesure de Radon sur $E = R^n$, $h(u, M)$ $(u \geqq 0, M \in E)$ une fonction continue de (u, M) décroissante en u, $f(K) = \int h(u_M, M)\, d\mu_M$ où $u_M =$ distance de M à K. (ii) Exemples de capacités d'ordre $\mathfrak{M}_\infty$: (1) μ est une mesure de Radon non négative dans E localement compact et $h(P, Q)$ est une fonction numérique continue non négative du couple (P, Q) $(P \neq Q)$, $f(K) = \int_{K^2} h(P, Q)\, d\mu_P\, d\mu_Q$ (supposée toujours finie), (2) dans le cadre de (i, 3), $f(K) =$ probabilité pour que pile ne se présente pas sur $E - K$. III—E étant localement compact, pour qu'une fonction f sur $\mathfrak{E} = \mathfrak{K}(E)$ soit la restriction d'une mesure de Radon positive, il faut et il suffit que f soit une capacité sur $\mathfrak{E}$ d'ordres $\mathfrak{A}_2$ et $\mathfrak{M}_2$ à la fois, finie et telle que $f(\phi) = 0$.

C. Y. Pauc.

Shanks, M. E. On the existence of measures. Revista Ci., Lima **53**, 31–40 (1951).

Let m be a finite outer measure in a space X. For any two elements of the space F of real-valued functions on X define

$$\Delta_m(f, g) = \inf \{y+m\}x;\ |f(x)-g(x)| \geqq y\};\ y>0\},$$

which is a translation-invariant pseudo-metric. Now let a pseudo-metric D on F be given, and define $M(A) = D(k_A, 0)$ where k_A is the characteristic function of the subset A of X. Conditions are developed which force M to be an outer measure; these conditions relate the convergence of sequences and that of their absolute values, and require a weak form of completeness. Supposing them satisfied, one can construct Δ_M. The main result is that then $D(f_n, 0)\to 0$ implies $\Delta_M(f_n, 0)\to 0$, but also $\Delta_M(f, 0) = 0$ implies $D(f, 0) = 0$.

R. Arens (Los Angeles, Calif.).

Marczewski, Edward. Measures in almost independent fields. Fund. Math. **38**, 217–229 (1951).

For the purposes of this paper a *measure* is a non-negative and *additive* set function μ defined on a Boolean *algebra* $\mathbf{M}$ of subsets of a set X and is such that $\mu(X) = 1$. An *extension* of a measure μ on $\mathbf{M}$ is a *measure* ν on an *algebra* containing $\mathbf{M}$ such that $\nu(E) = \mu(E)$ whenever $E \in \mathbf{M}$. The definitions of the concepts of σ-measure and σ-extension are obtained from the preceding sentences by qualifying each italicized word with the adjective σ. A family $\{\mathbf{M}_t\}$ of algebras carrying a corresponding family $\{\mu_t\}$ of measures is almost-independent (almost-σ-independent) if, for each finite (countable) sequence $\{t_n\}$ of distinct indices, $\bigcap_n E_n \neq 0$ whenever $E_n \in \mathbf{M}_{t_n}$ and $\mu_{t_n}(E_n) \neq 0$ for all n. A common extension μ of all μ_t is multiplicative if $\mu(\bigcap_n E_n) = \prod_n \mu(E_n)$ whenever $E_n \in \mathbf{M}_{t_n}$.

The author's main results are that (1) a common multiplicative extension of the measures μ_t exists (and is then necessarily unique) if and only if the algebras $\mathbf{M}_t$ are almost-independent, and (2) if each μ_t is a σ-measure, if all the σ-algebras $\mathbf{M}_t$, with at most one exception, are finite, and if the family $\{\mathbf{M}_t\}$ is almost-σ-independent, then there exists a common multiplicative σ-extension. *P. R. Halmos.*

Blau, J. H. The space of measures on a given set. Fund. Math. **38**, 23–34 (1951).

A measure in this paper is a non-negative, normalized, monotone, countably subadditive, and regular set function defined on the class of all subsets of a T_1 space R, and is such that all open sets are Carathéodory measurable. (In fact the author begins by considering structures R more general than topological spaces, but, not being very profitable, the extra generality is soon abandoned.) A topology is defined in the set M of measures by the subbasic neighborhoods $\{\varphi: \varphi \in M,\ \varphi_0(U) < \varphi(U)+a\}$, where $\varphi_0 \in M$, U is open, and $a>0$. The following results are typical. (1) M is a T_1 space. (2) If R has a countable base, then so does M. (3) M is a compactum if and only if R is. (4) If R is normal, then convergence in M is equivalent to weak convergence.

P. R. Halmos (Chicago, Ill.).

Caccioppoli, Renato. Misura e integrazione sugli insiemi dimensionalmente orientati. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 3–11 (1952).

The author initiates in this note (the first of a series) a brief sketch of what he calls a general theory of integration on a k-dimensional variety in Euclidean space of $n>k$ dimensions. The work of others on this topic [e.g. Whitney, Proc. Nat. Acad. Sci. U. S. A. **33**, 1–6 (1947); these Rev. **8**, 397], work which goes deeply into the topological background, has little or no connection with the author's train of thought, and is unknown to him. If the author's theory is of comparable scope or depth, this is not made clear in the note, which restricts itself "as an illustration" to the case $n=3$, $k=2$ and treats the intersection of Borel sets with countable sums of boundaries of rather special domains, to which a positive direction of the normal is assigned.

L. C. Young (Madison, Wis.).

Erdös, P. A theorem on the Riemann integral. Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 142–144 (1952).

The author shows that if $f(x)$ is a real function such that, for all h, $\int |f(x+h)-f(x)|\,dx = 0$, then, for some constant c, $\int |f(x)-c|\,dx = 0$; the integrals are Riemann integrals over $(-\infty, \infty)$. [Cf. the following review.] *R. P. Boas, Jr.*

de Bruijn, N. G. A difference property for Riemann integrable functions and for some similar classes of functions. Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 145–151 (1952).

In a previous paper [Nieuw Arch. Wiskunde (2) **23**, 194–218 (1951); these Rev. **13**, 332] the author showed that several classes C of functions have the property that if $f(x+h)-f(x)$ is in C for all h, then $f(x) = g(x)+H(x)$, where $g(x)$ is in C and $H(x+y) = H(x)+H(y)$. Here he defines some quite general classes of functions and shows that they have the property in question, thus achieving a rather systematic approach to such questions. The classes considered include some cases left open before and allow simpler proofs of some of the cases treated previously. In particular, the property is proved when C consists of Riemann integrable functions [cf. the preceding review], continuous functions, functions of bounded variation, functions with bounded derivative, or trigonometric polynomials. *R. P. Boas, Jr.*

Hill, J. D. A note on indefinite integrals. Proc. Amer. Math. Soc. **3**, 263–269 (1952).

Let $f(x)$ be a real-valued, Lebesgue integrable function in the closed interval $[0, 1]$, E be any measurable subset of this interval, $F(E) = \int_E f(x)dx$ and $B_*(\alpha)$, $B^*(\alpha)$ be the lower and upper bounds of $F(E)$ taken over all sets of measure α. By studying the properties of $B_*(\alpha)$, $B^*(\alpha)$ and using functions obtained from $f(x)$ by a rearrangement the author shows that the values of $F(E)$ for $|E| = \alpha$ comprise the whole of the closed interval $[B_*(\alpha), B^*(\alpha)]$. He also remarks on the possibility of assigning a mean value for the set of all $F(E)$ as E varies over all measurable subsets of $[0, 1]$.

H. G. Eggleston (Swansea).

Walters, S. S. On Ascoli's theorem. Amer. Math. Monthly **59**, 237–238 (1952).

The author gives an elementary example of a sequence of continuous and uniformly bounded functions $\{f_n(x)\}$ in $[0, 1]$, which converges on a denumerable dense subset of $[0, 1]$ and diverges at the other points of $[0, 1]$. [The author was not aware that another example of such a type had already been given by F. Hausdorff, Grundzüge der Mengenlehre, Veit, Leipzig, 1914, p. 397; see also Mengenlehre, 2nd ed., de Gruyter, Berlin-Leipzig, 1927, pp. 270–272.]

A. Rosenthal (Lafayette, Ind.).

Scorza Dragoni, Giuseppe. Una applicazione della quasi-continuità semiregolare delle funzioni misurabili rispetto ad una e continue rispetto ad un'altra variabile. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 55–61 (1952).

Consider, in the xy-plane, the strip S:

$$a \leqq x \leqq b, \quad -\infty < y < +\infty.$$

Let $f(x, y)$ be a function which satisfies in S the following conditions. (a) f is measurable with respect to x and continuous with respect to y. (b) f is dominated in S by a summable function of x alone. It is known that under these conditions the differential equation $y' = f(x, y)$ has solutions, in I: $a \leqq x \leqq b$, in the following sense: there exist in I absolutely continuous functions $g(x)$ such that $g'(x) = f(x, g(x))$ in I, with the possible exception of a set of measure zero $e(g)$, depending upon the particular solution g. The purpose of this paper is to show that the union of all these exceptional sets $e(g)$, corresponding to the various solutions g, is still a set of measure zero. In other words, there exists a subset E of I such that $I - E$ is of measure zero and every absolutely continuous solution g (in the sense explained above) is differentiable and satisfies the differential equation on the set E. The proof follows quite readily from previous results of the author in case $f(x, y)$ is bounded in S. The general case requires some additional study. *T. Radó.*

Cesari, Lamberto. Sul teorema di derivazione delle funzioni a variazione limitata in un insieme. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) **8** (1950–51), 223–229 (1952).

The author supplements his previous results [same Mem. (10) **2**, 127–145 (1946); **3**, 99–112 (1947); these Rev. **10**, 288] concerning continuous functions $f(x)$ of bounded variation (b.v.) on a subset E of the interval $a \leqq x \leqq b$. He writes $V(E, f) = \sup \sum_{i=1}^{n} |f(x_{i+1}) - f(x_i)|$ for all possible groups $a \leqq x_1 < x_2 < \cdots < x_n \leqq b$ of points of E, and

$$W(E, f) = \int_{-\infty}^{\infty} N(E, y)\,dy,$$

where $N(E, y)$ is the number of roots in E of $f(x) = y$. He calls $f(x)$ of b.v. $V[W]$ when $V(E, f)$ $[W(E, f)]$ is finite; in the latter case E is to be a Borel set, so that $N(E, y)$ is measurable. If $f(x)$ is of b.v. V (VB in Saks' terminology) on E then it has an approximate derivative almost everywhere in E by the Denjoy-Hinčin theorem [cf. S. Saks, Theory of the integral, Warsaw, 1937, p. 222]. In contrast to this the author constructs an example of a function continuous on I: $0 \leqq x \leqq 1$, of b.v. W on a perfect set $E \subset I$ of positive measure, and not possessing an approximate derivative (finite or infinite) at any point of E.

H. P. Mulholland (Birmingham).

Baiada, Emilio. L'area delle superficie armoniche quale funzione delle rappresentazioni del contorno. Rivista Mat. Univ. Parma **2**, 315–330 (1951).

Partial answer to a question raised by Morse and Tompkins [Amer. J. Math. **63**, 825–838 (1941); these Rev. **3**, 249]. A complete answer was given by K. H. Carlson and the reviewer [Proc. Amer. Math. Soc. **3**, 88–91 (1952); these Rev. **13**, 731]. *L. C. Young* (Madison, Wis.).

Besicovitch, A. S. On two problems of Loewner. J. London Math. Soc. **27**, 141–144 (1952).

The following definitions are needed. A topological square is a set of the form $E = T(Q)$, where Q is a Euclidean square, and T is a topological mapping. The vertices of E are, by definition, the images under T of the vertices of Q. It is assumed that E lies in Euclidean three-space.—Given a set F in three-space, and two points p, q on F, their S-distance is defined as the infimum of the lengths of all arcs joining p and q in F. The definition of the S-distance of two subsets of F follows in the customary manner.—Now let E be a topological square, with vertices v_1, v_2, v_3, v_4, and let a and h denote the S-distances of the pairs of opposite sides v_1v_2, v_3v_4 and v_2v_3, v_4v_1 respectively. The author shows that the area of E is $\geqq ah$. Actually, by means of elementary considerations, he first obtains the conclusion that the Lebesgue area $L(E)$ of E is $\geqq ah$. Since $L(E)$ is at most equal to $H^2(E)$, the two-dimensional Hausdorff measure of E, it follows that $H^2(E) \geqq ah$. For the special case when E is a subset of a plane, the result is due to Loewner. The paper contains an explicit example showing that an analogous equality relating to topological cylinders, conjectured by Loewner, is generally not satisfied. *T. Radó* (Columbus, Ohio).

Besicovitch, Abram S. Definition of the area of a surface. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **10**, 135–139 (1951).

This paper is the text of an expository lecture on the work of the author in surface area theory. The general objective of this work is the construction of a theory based on the Hausdorff-Carathéodory scale of measures of various dimensions. For details, see the accounts on the work of the author in these Rev. **10**, 520, 521; **7**, 282 in particular.

T. Radó (Columbus, Ohio).

Federer, Herbert. Hausdorff measure and Lebesgue area. Proc. Nat. Acad. Sci. U. S. A. **37**, 90–94 (1951).

Introduction from the paper: "Following the creation of modern integration theory by Lebesgue and Carathéodory, it was natural to extend the concept of area to arbitrary continuous surfaces. This was done in several intuitively satisfactory ways, each offering its own geometric and analytic interest. It is now possible to associate with every surface several numbers, each of which claims a good right to be called the area of the surface, but the relationship between these numbers is far from obvious in the general case. Efforts to determine which equations or inequalities are satisfied by these several areas of a surface have greatly added to our knowledge of significant metric and topological properties of continuous surfaces. Thus the lack of uniqueness in the definition of surface area has inspired the creation of a comparative theory, whose main results are summarized in this note. Some of these results are explicitly or implicitly contained in the literature, while others are published here for the first time." The paper is so concise that a listing of results, preceded by pertinent definitions, would be almost as long as the paper itself. Two minor misprints: In (14) and (16), $H_{n,n}$ should be $H_{n+1,n}$. *J. F. Randolph.*

Theory of Functions of Complex Variables

***Tricomi, Francesco. Funzioni analitiche.** Ristampa della seconda edizione. Nicola Zanichelli Editore, Bologna, 1952. vii+134 pp. 1500 lire.

See these Rev. **8**, 200 for a review of the 1st printing of this edition.

Pradillo, Julio Garcia. An observation on the conditions of monogeneity. Gaceta Mat. (1) **3**, 179–182 (1951). (Spanish)

The author studies several specific examples to show that functions $U(x)$ and $V(x)$ may satisfy the Cauchy-Riemann equations, as well as Laplace's equation, without $U+iV$ being monogenic, and conversely. The examples are of the form $U=\partial f/\partial x$, $V=\partial f/\partial y$, where $f(x)=y^4 \sin(x/y)$ in one case, and $f(x)=(x^2+y^2)\arctan(x/y)-\frac{1}{3}xy^2$ in the other.

R. C. Buck (Madison, Wis.).

Cowling, V. F. On the distribution of the values of the partial sums of a Taylor series. Proc. Amer. Math. Soc. **2**, 732–738 (1951).

Mit Hilfe einer von Leighton und Thron [Duke Math. J. **9**, 763–772 (1942); diese Rev. **4**, 195] entwickelten Methode gibt Verf. für die Koeffizienten einer Potenzreihe in Verbindung mit einem vorgeschriebenem Variabilitätsbereich von z hinreichende Bedingungen an, damit die Werte der Partialsummen $S_n(z)=1+a_1z+\cdots+a_nz^n$ einem gewissen Bereich angehören. Der folgende Satz ist für die erhaltenen Resultate typisch: Liegen die Werte $a_na_{n-1}^{-1}z$ für ein p mit $0<p\leqq 1$ und $z\in Z$ in dem Bereich

$$r\geqq[(p^2-1)/2p]\cos\theta+(p+1)^2/2p,\quad 0<\theta\leqq 2\pi,$$

so gilt $|S_n(z)-\frac{1}{2}(1-p)|\geqq\frac{1}{2}(1-p)$ für alle n.

A. Pfluger (Zürich).

Yu, Chia-Yung. On some functions holomorphic in an infinite region. Proc. Amer. Math. Soc. **3**, 232–236 (1952).

Starting from a result proved by the reviewer [Ann. Sci. Ecole Norm. Sup. (3) **63**, 351–378 (1947); these Rev. **9**, 229, 735], the author indicates conditions on the function $g(x)$, the sequence $\{\nu_n\zeta\}$ and the growth at infinity of the function $F(z)$, holomorphic in the region $x>d$ $(d<0)$, $|y|\leqq g(x)$, and satisfying the condition $F^{(\nu_n)}(0)=0$, in order that $F(z)\equiv 0$.

S. Mandelbrojt (Houston, Texas).

Ilieff, Ljubomir. Über die Abschnitte der schlichten Funktionen, die den Kreis $|z|<1$ konvex abbilden. Annuaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1. **46**, 153–159 (1950). (Bulgarian. German summary)

It is proved that if $f_k(z)=z+\sum_1^\infty a_\nu^{(k)}z^{\nu k+1}$ is univalent, k-wise symmetric, and convex in $|z|<1$, then the sum of the first $n+1$ terms is univalent in

$$|z|<\left(1-\frac{k}{k+1}\left(1+\frac{2}{k}\right)\frac{\ln(n+1)}{n+1}\right)^{1/k},$$

for $n+1>\exp k(2k)^{1/2}/(2+k)$, for $k=1, 2, \cdots$.

A. W. Goodman (Lexington, Ky.).

Ilieff, Ljubomir. Sätze über die Abschnitte der schlichten Funktionen. Annuaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1. **46**, 147–151 (1950). (Bulgarian. German summary)

Translation into Bulgarian of an article reviewed previously [Acta Math. Acad. Sci. Hungar. **2**, 109–112 (1951); these Rev. **13**, 640]. *A. W. Goodman* (Lexington, Ky.).

Ilieff, Ljubomir. Über die 3-symetrischen schlichten Funktionen. Annuaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1. **46**, 161–165 (1950). (Bulgarian. German summary)

Translation into Bulgarian of an article reviewed previously [Doklady Akad. Nauk SSSR **79**, 9–11 (1951); these Rev. **13**, 123] *A. W. Goodman* (Lexington, Ky.).

Iliev, Lyubomir. Theorems on triply symmetric univalent functions. Doklady Akad. Nauk SSSR (N.S.) **84**, 9–12 (1952). (Russian)

Let $f(z)=z+\sum_1^\infty a_\nu z^{3\nu+1}$ be univalent in $|z|<1$. It is proved that the partial sums are univalent in $|z|<\frac{1}{3}\sqrt{3}$. Improved bounds are obtained for $|a_\nu|$, $\nu=3, 4, \cdots, 9$.

A. W. Goodman (Lexington, Ky.).

Sinclair, Annette. Generalization of Runge's theorem to approximation by analytic functions. Trans. Amer. Math. Soc. **72**, 148–164 (1952).

Let S be a closed point set of the extended plane, $I_j(S)$ the open regions (i.e. connected sets) into which S divides the plane, and let the points $z_j\in I_j(S)$ be arbitrarily chosen $(j=1, 2, \cdots)$. Every function $M(z)$ analytic on S can be approximated uniformly by rational functions whose poles lie in the z_j. For this and related results see J. L. Walsh, Interpolation and approximation . . . [Amer. Math. Soc. Colloq. Publ., vol. 20, New York, 1935], ch. 1, theorems 8, 9, 16, 17.

The author considers, in substance, sets S which are not closed but consist of an infinity of closed components $S_j(j=1, 2, \cdots)$; while the components $I_j(S)$ of the complementary set $C(S)$ may be finite or infinite in number. For instance, S is an infinity of congruent circular or annular closed regions, mutually disjoint; or of nested rings which do not abut. A point that is a limit point of some set of points $z_j(z_j\in S_j$ arbitrarily chosen, $j=1, 2, \cdots)$ is called a sequential limit point; the set formed by all those limit points is denoted by B. When $B\subset C(S)$ then S is a "Q-set". Any set consisting of B and precisely one point of each $I_j(S)$ such that $I_j(S)\cap B=0$ is "a $B^*(S)$ set". The set B^*-B plays a similar part in the results as the set $\{z_j\}$ in the Walsh theorems. When B does not divide the plane the approximating functions $L(z)$ are monogenic; while otherwise, as the reviewer remarks, S and $M(z)$ can be constructed so that $L(z)$ necessarily consists of an infinity of different analytic functions.

Continuing work done by P. W. Ketchum [same Trans. **49**, 211–228 (1941); these Rev. **2**, 273], the author first establishes a number of topological results, of interest in themselves; and then proves results of the following kind (see Theorems 3, 4, 5). Let $S=\sum S_j$ be a Q-set, and a $B^*(S)$ be fixed. If $M(z)$ is analytic on S then, given $\delta_j>0$, there is a $L(z)$ analytic in $C(B^*)$, meromorphic in $C(B)$ (i.e. with poles on B^*-B), and such that $|M(z)-L(z)|<\delta_j$ on $S_j(j=1, 2, \cdots)$. When E is a set such that $B\subset E\subset C(S)$, every function analytic on S can be approximated by functions analytic in $C(E)$ if and only if every $I_j(S)$ contains a point of E. If $M(z)$ is $\neq 0$ and analytic on S, $\delta_j>0$, and if $\log M(z)$ can be chosen as single valued and continuous on S, then there is a $L(z)$, $\neq 0$ and analytic in $C(B^*)$, such that $|M(z)-L(z)|<\delta_j$ $(z\in S_j, j=1, 2, \cdots)$. Then a theorem by Weierstrass is extended. So is the Mittag-Leffler partial-fractions theorem: if S is a Q-set, $\delta_j>0$, and $M(z)$ meromorphic on S, then there is a $L(z)$ which is analytic in $C(B)$ except exactly at the poles of $M(z)$ on S, and is such that, except for those poles, $|M(z)-L(z)|<\delta_j$ $(z\in S_j, j=1, 2, \cdots)$. Finally a result is deduced on the case when $L(z)$ is an entire function of finite order.

H. Kober (Birmingham).

Lohin, I. F. On functions representable by series of Dirichlet polynomials. Doklady Akad. Nauk SSSR (N.S.) **83**, 13–15 (1952). (Russian)

The author proves the following theorem: A necessary and sufficient condition that $F(\zeta)$, holomorphic for $R_\zeta\geqq 0$

and of exponential type, be a limit of expressions of the form $P_n(\zeta)=\sum_{\nu=1}^{n}a_{n\nu}e^{-\lambda_\nu\zeta}$, $0<\lambda_1<\lambda_2\cdots$, with $\lim n/\lambda_n=\sigma<\infty$, these expressions converging uniformly in each bounded region contained in $\Re\zeta\geqq\gamma>0$, is that the function

$$f(\zeta)=\int_0^\infty F(z)e^{-z\zeta}dz$$

be meromorphic with simple poles at $-\lambda_\nu(\nu\geqq 1)$. To prove his theorem the author studies a certain type of solutions of an equation of the form $\int f(\zeta)G(\zeta)e^{z\zeta}d\zeta=0$, the integral being extended over an arc, $G(\zeta)$ being a given entire function of exponential type. *S. Mandelbrojt.*

*Fuks, B. A. **Neevklidova geometriya v teorii konformnyh i psevdokonformnyh otobraženiĭ.** [**Non-Euclidean geometry in the theory of conformal and pseudo-conformal mappings.**] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951. 148 pp. 5.20 rubles.

The present work is one of a series of small volumes, each devoted to an exposition of some aspect of non-Euclidean geometry. The author's principal emphasis is on Bergman's theory of the invariant metric and the relation of this theory to the classical portions of non-Euclidean geometry and conformal mapping. The material is self-contained, and is well-developed in the author's usual lucid style. However, the work suffers considerably because of the conspicuous absence of new material which has been developed over the past ten years by numerous writers and which serves to relate in a striking manner the theory of the kernel function and the invariant metric with potential theory [cf. Bergman, The kernel function and conformal mapping, Amer. Math. Soc., New York, 1950; these Rev. 12, 402].

Chapter 1 presents the development of Bergman's kernel function through the use of complete orthonormal complex analytic systems. The invariant metric is introduced, and the generalized lemma of Schwarz-Pick is proved. In chapter 2, the author establishes several differential-geometric characterizations of the invariant metric, and then turns to classical questions of Lobatchewskian geometry: groups of motions, distances, axioms of congruence, parallelism. Chapter 3 considers the geometry of automorphic functions. In chapter 4, the study of the lemma of Schwarz-Pick is resumed. Applications are made to Julia's distortion theorem. In chapter 5, the author turns to pseudo-conformal transformations of the space of two complex variables. The kernel function $K_D(z_1, z_2; \bar{z}_1, \bar{z}_2)$ is introduced, and by its means, Bergman's invariant metric

$$ds^2=\sum_{m,n=1}^{2}T_{mn}dz_m d\bar{z}_n;\quad T_{mn}=\frac{\partial^2\log K_D}{\partial z_m\partial\bar{z}_n}.$$

Finally, questions of curvature and infinitesimal transformations are discussed. *P. Davis.*

Mazza, Giuseppina. Sopra la rappresentazione di una funzione olomorfa in una regione ellittica. Ricerca, Napoli 2, no. 2, 60–64; no. 3–4, 27–32 (1951).

It is shown that a function $f(z)$, analytic in the elliptical region $x^2/a^2+y^2/b^2<1$, can be expanded there into a series of polynomials in the two variables z and $(z^2-c^2)^{1/2}$, where $c^2=a^2-b^2$. The principal innovation occurs in the algebraic character of the proof. *A. J. Lohwater.*

Tietz, Horst. Fabersche Entwicklungen auf geschlossenen Riemannschen Flächen. J. Reine Angew. Math. 190, 22–33 (1952).

By expanding a properly normalized elementary differential of the third kind $dF(y, z)$ on a closed Riemann surface A in a power series in the local coordinates, the author obtains a complete system of elementary functions $E_n(z)$ and elementary differentials $dF_n(y)$ on A which play a role analogous to the Faber polynomials in plane regions. If G is a region on A bounded by an analytic curve and having a simply connected complementary region, any regular, single-valued function $f(z)$ [differential $dg(y)$] in G can be expanded in a series $f(z)=\sum a_nE_n(z)$ [$dg(y)=\sum b_ndF_n(y)$] convergent in G. *G. Springer* (Evanston, Ill.).

Künzi, Hans P. Surfaces de Riemann avec un nombre fini d'extrémités simplement et doublement périodiques. C. R. Acad. Sci. Paris 234, 1660–1662 (1952).

In a recent paper [same C. R. 234, 793–795 (1952); these Rev. 13, 643], the author was concerned with Riemann surfaces with a finite number of doubly periodic ends. In the present paper, he extends the treatment to the case of a finite number of singly and doubly periodic ends. The following general theorem is announced. For a surface with a finite number of singly and doubly periodic ends, there are no exceptional values of Nevanlinna. *L. Sario.*

Fourès, Léonce. Sur les recouvrements régulièrement ramifiés. Bull. Sci. Math. (2) 76, 17–32 (1952).

Closed covering surfaces $\bar{R}$ of a closed Riemann surface R are considered. The author defines $\bar{R}$ to be regularly ramified over a point ω of R if all points of $\bar{R}$ over ω are branch points of the same order ν. He obtains the following complete result. Let n ($\geqq 2$) be a finite number of points $\omega_i\varepsilon R$ and ν_i arbitrary natural numbers. If the genus of R is $\geqq 1$, then there always exists a closed $\bar{R}$ which is regularly ramified of order ν_i over ω_i and has no other branch points. This remains true for a planar R, except if $n=2$ and $\nu_1\neq\nu_2$. The proof is given by a study of the topological tree of $\bar{R}$. *L. Sario* (Stanford University, Calif.).

Noshiro, Kiyoshi. Open Riemann surface with null boundary. Nagoya Math. J. 3, 73–79 (1951).

Es wird der Gross'sche Satz über die Umkehrfunktion einer ganzen Funktion folgendermassen verallgemeinert: Wenn die zum Funktionselement $f=a_0+a_1z+\cdots$ gehörige Riemannsche Fläche im Nevanlinnaschen Sinne nullberandet ist, so kannd as Funktionselement f längs aller Strahlen $\arg z=\varphi_1$, $0\leqq|z|<\infty$, fortgesetzt werden mit Ausnahme höchstens einer Menge vom Mass null. Dies gelingt mit der klassischen Methode von Gross unter Verwendung der Tatsache, dass die Riemannsche Fläche F derart durch eine wachsende Folge von kompakten Teilflächen F_n ausgeschöpft werden kann, dass die Summe $\sum^n\mu_n$ der Moduln von $F_n-\bar{F}_{n-1}$ divergiert. Aus dem gleichen Grunde folgt, das eine solche nullberandete Fläche im Ahlforsschen Sinne [Acta Math. 65, 157–194 (1935)] regulär ausschöpfbar ist [vgl. auch K. Noshiro, Jap. J. Math. 19, no. 4, 299–327 (1948); diese Rev. 11, 428]. Es kann also die Ahlforssche Theorie auf diese allgemeinere Flächenklasse übertragen werden. Da zwei Riemannsche Flächen, die aufeinander quasikonform abgebildet werden können, gleichzeitig nullberandet sind, so bleibt die Gültigkeit der Ahlforsschen Theorie auch für pseudoanalytische Funktionen auf einer nullberandeten Fläche bestehen. *A. Pfluger* (Zürich).

Kuroda, Tadashi. Some remarks on an open Riemann surface with null boundary. Tôhoku Math. J. (2) 3, 182–186 (1951).

Let $\{F_n\}$ be an exhaustion of an arbitrary open Riemann surface F and let Γ_n be the boundary of F_n. In $F_{n+1}-F_n$,

let x be the harmonic function, determined by $x=0$ on Γ_n, $x=\log\sigma_n=\text{const.}$ (>0) on Γ_{n+1}, such that $\int_{\Gamma_n}d\bar{x}=2\pi$. The author considers the following type criterion of the reviewer [C. R. Acad. Sci. Paris 230, 269–271 (1950); these Rev. 11, 342]: If there exists such an exhaustion $\{F_n\}$ that $\prod\sigma_n$ diverges, then the surface is of parabolic type. Noshiro [in the paper reviewed above] proved that this condition is also necessary. The author reproves Noshiro's result. He also re-establishes that parabolic type implies the non-existence of harmonic bounded or harmonic "Dirichlet bounded" functions.

The author refers to the interesting question about the origin of "nullbounded" surfaces. The historical background is as follows. By his mapping theorem, based upon the existence or non-existence of the Green's function, Riemann introduced the division of open surfaces into two types which are at present termed parabolic and hyperbolic. In the last section of his inaugural dissertation [Göttingen, 1851], Riemann specifically states that his reasoning is not restricted to the case of simply connected surfaces. The notion (not the name) "harmonic measure" was introduced by Schwarz in 1890 on p. 360 of his Gesammelte Mathematische Abhandlungen, vol. II [Springer, Berlin]. There $\chi/2\pi$ is the harmonic measure, used for giving to the Poisson integral the elegant form $(2\pi)^{-1}\int f(\vartheta)d\chi$. The current notation $\omega(z, \gamma, D)$ for the harmonic measure was introduced by Beurling in his dissertation [Upsala, 1933] on p. 26. A year after this extensive investigation on the subject, ω was named "harmonic measure" by Nevanlinna [Åttonde skandinaviska matematikerkongressen, Stockholm, 1934, Ohlsson, Lund, 1935, pp. 116–133]. The later application of Schwarz-Beurling's harmonic measure for defining the "null-bounded" surfaces repeats Riemann's classification [cf. Sario, C. R. Acad. Sci. Paris 230, 42–44 (1950); these Rev. 11, 342]. This equivalence is also reproved in the present paper.

L. Sario (Stanford University, Calif.).

Lelong, P. Sur les séries de Taylor $F(x, y)$ ayant des coefficients entiers. Publ. Math. Debrecen 1, 209–221 (1950).

The author considers questions relating to the rationality of analytic functions of two complex variables whose Taylor coefficients are integers. For Taylor series in one variable such questions were previously considered by Borel and by Pólya [Math. Ann. 99, 687–706 (1928)]. Let D be a domain in the space of two complex variables x and y and containing the origin 0. Let $E(D)$ designate the class of functions $F(x, y)$ which are analytic and single-valued in D and all of whose Taylor coefficients $a_{pq}=(p!q!)^{-1}(\partial^{p+q}F/\partial x^p\partial y^q)_0$ are integers. The author gives certain sufficient conditions on D in order that $E(D)$ contain only rational functions. This result is applied to the case in which D is a Hartogs domain of regularity H defined by H: $[y\in d, |x|<R(y)]$. Here d designates a domain lying in the complex y-plane and $-\log R(y)$ is subharmonic or $-\infty$. It is shown that a necessary and sufficient condition that $E(H)$ contain only rational functions is that (a) there exists an open set e containing the origin and contained in $[x=0, y\in d]$ on which $R(y)>\rho>1$, (b) under the transformation $y'=y^{-1}$, the component e_0 of e which contains the origin goes into a region whose closed bounded complement σ possesses a transfinite diameter $c(\sigma)<1$.

P. Davis (Washington, D. C.).

Saxer, Walter. Sur les domaines de normalité des fonctions méromorphes de plusieurs variables. J. Math. Pures Appl. (9) 31, 49–53 (1952).

If B is a schlicht bounded domain of meromorphy for functions of two complex variables, if a family of functions is meromorphic in B and if there exists a subdomain B^* of B having the property that the family is normal in B^*, then the author has called the largest such domain B^* a domain of normality. In 1932 [Comment. Math. Helv. 4, 256–267 (1932)] he showed that domains of normality must satisfy the same boundary condition (Levi's condition on pseudoconvexity) as domains of regularity and meromorphy, if in the interior of the domain in question the family possesses non-essential irregular points. In the present paper, using a fundamental theorem of K. Oka [Tôhoku Math. J. 49, 15–52 (1942); these Rev. 7, 290], the author shows that every finite schlicht pseudoconvex domain in the space of two complex variables can be considered as a domain of normality for a family of holomorphic or meromorphic functions. The author concludes the paper with a reference to a letter from H. Behnke stating that Oka's result has been carried over by Hitotumatu and by Bremermann to n complex variables $(n\geqq 2)$ and the author states that the present result can similarly be carried over. [The author wishes to call attention to errors which were made in the translation of his paper from German to French. On page 49, lines 7, 17, and 20 read "de méromorphie' for "d'holomorphie"; line 12, read "normale" for "régulière"; page 52, line 10, read "de transformer" for "de" and "en" for "d'extraire"]. *W. T. Martin* (Cambridge, Mass.).

Bochner, S. On the addition theorem for multiply periodic functions. Proc. Amer. Math. Soc. 3, 99–106 (1952).

L'A. osserva che il teorema d'addizione della teoria delle funzioni ellittiche o più in generale abeliane non è in realtà un teorema, che poggi essenzialmente sopra la nozione di additività; esso può concepirsi infatti come un caso molto particolare di teoremi "locali", nei quali l'additività è completamente assente. I teoremi "locali" di cui trattasi sono i seguenti: 1) Se A è un dominio nello spazio delle k variabili complesse $z=(z^{(1)}, \cdots, z^{(k)})$ e B un dominio nello spazio delle l variabili complesse $w=(w^{(1)}, \cdots, w^{(l)})$; se è dato un numero finito di funzioni olomorfe $\varphi^i(z)$ $(i=1, 2, \cdots, r)$ in A ed un numero finito di funzioni olomorfe $\psi^j(w)$ $(j=1, 2, \cdots, s)$ in B; se la funzione olomorfa $f(z, w)$ in $A\times B$ è funzione algebrica delle $\varphi^i(z)$ per ogni w in B e funzione algebrica delle $\psi^j(w)$ per ogni z in A, allora la $f(z, w)$ è funzione algebrica delle $\varphi^i(z)$ e delle $\psi^j(w)$ in $A\times B$; 2) se $f(z, w)$ è funzione razionale delle $\varphi^i(z)$ per ogni w in B e funzione razionale delle $\psi^j(w)$ per ogni z in A, allora la $f(z, w)$ è funzione razionale delle $\varphi^i(z)$ e delle $\psi^j(w)$ in $A\times B$; 3) se $f(z, w)$ è funzione algebrica delle $\varphi^i(z)$ per ogni w in B di grado non superiore a g, con g indipendente da w, e se $f(z, w)$ è funzione razionale delle $\psi^j(w)$ per ogni z in A, allora la $f(z, w)$ è funzione algebrica delle $\varphi^i(z)$ e delle $\psi^j(w)$ di grado non superiore a g. La dimostrazione di tali teoremi poggia sul lemma seguente: Se $F_h(z, w)$ $(h=1, 2, \cdots, N)$ sono funzioni in $A\times B$ non identicamente nulle e soddisfacenti ad una relazione $\sum_1^N c_h(w)F_h(z, w)=0$, con le $c_h(w)$ non simultaneamente nulle per ciascun w in B, allora le $F_h(z, w)$ soddisfano ad una relazione dello stesso tipo dove le $c_h(w)$ sono sostituite da funzioni $C_h(w)$, non tutte nulle identicamente ed appartenenti al più piccolo anello sopra il corpo complesso, che contiene tutte le funzioni $F_h(a, w)$ per tutti i valori particolari di $z=a$ in A ed $h=1, 2, \cdots, N$.

Si osservi che in questo lemma non si parla affatto di analiticità e nemmeno di continuità delle funzioni considerate.

Il teorema d'addizione della teoria delle funzioni abeliane, benchè conseguenza dei teoremi anzidetti, si distingue tuttavia per il fatto che in esso le funzioni in $A \times B$ da considerarsi sono funzioni della $z+w$. L'A. svolge qualche considerazione ed indica qualche generalizzazione di questa circostanza. *F. Conforto* (Roma).

Ozaki, Shigeo, Ono, Isao, and Ozawa, Mitsuru. A theorem of the pseudo-meromorphic function. Sci. Rep. Tokyo Bunrika Daigaku. Sect. A. 4, 203–205 (1951).

The authors undertake to prove a theorem of MH (Morse and Heins) [Ann. of Math. (2) 46, 625–666 (1945); these Rev. 8, 21] concerning a mapping $w=f(z)$ by a pseudo-meromorphic function f from a finitely connected schlicht domain D whose boundary Γ consists of regular curves, into the w-plane. The MH relation $\mu-2n(\infty)+2-\nu=p$ is affirmed to exist under the same conditions as by MH, with two exceptions. The angular order of a boundary image g is given both in the theorem and proof by an integral which is not defined if g is merely locally simple as with MH. The essential hypothesis that f be locally $1-1$ in a neighborhood of each point Γ is omitted so that the proof cannot be valid without reading in this hypothesis. There is no proof that the number of ramification points is finite. To omit the hypothesis of local 1—1-ness near points of Γ can be compensated for by defining ramification orders of boundary points as first done by Morse [Topological methods in the theory of functions of a complex variable, Princeton Univ. Press, 1947; these Rev. 9, 20]. *M. Morse.*

Ozaki, Shigeo, Ono, Isao, and Ozawa, Mitsuru. Second principal theorem of the pseudo-meromorphic mapping in the three dimensional space. Sci. Rep. Tokyo Bunrika Daigaku. Sect. A. 4, 207–210 (1951).

The authors are concerned with a real single-valued mapping of a subset D of a euclidean 3-space E into E conventionally closed by a point at infinity. The domain D is bounded by a finite number of regular surfaces of finite genus. The poles of f are to accumulate only on the boundary Γ of D. The mapping is of class C' with a jacobian which is non-vanishing except on a countable set of "ramification lines" (r.l.) which accumulate only on Γ. If the reviewer interprets this abstract correctly the r.l. are simple, may be either closed or pass from Γ to Γ, and are unknotted and mutually non-linking. Then it is affirmed that $2-\nu-\lambda+\mu-2n(\infty)=p$, generalizing results of Morse and Heins in plane topological function theory [Ann. of Math. (2) 46, 625–666 (1945); these Rev. 8, 21]. Here ν is the number of components in Γ, λ the genus of Γ, μ the sum of the ramification orders of open r.l., $n(\infty)$ the sum of multiplicities of poles of f suitably counted, p the algebraic number of times a direction sphere is covered in the mapping onto the direction sphere induced by the normal to the image of Γ. It is hoped that a more detailed account will clarify the meaning and proof of this interesting theorem. *M. Morse* (Princeton, N. J.).

Theory of Series

Ribeiro de Albuquerque, José. Series with arbitrary terms. Gaz. Mat., Lisboa 12, no. 50, 29–34 (1951). (Portuguese)

Let $\alpha_0+\alpha_1+\alpha_2+\cdots$ be a real series containing a section with p_0 positive terms, then a section with p_1 negative terms, then a section with p_2 positive terms, and so on, where $p_n>0$. The series $a_0-a_1+a_2-\cdots$, in which a_n is the sum of the absolute values of the terms in the section with index n, is called the equivalent alternating series. The author undertakes to develop comparisons of the behaviors of series and their equivalent alternating series, and to give new theorems on alternating series. The effort is marred by mistakes. *R. P. Agnew* (Ithaca, N. Y.).

Szász, Otto. On products of summability methods. Proc. Amer. Math. Soc. 3, 257–263 (1952).

Let A and B be two regular summability methods for sequences s_n and AB denote the A transform of the B transform of the sequence s_n. The author shows that A summability implies AB summability to the same sum when A is the Abel, Laplace or Borel transform and B is the Cesaro mean of order α; and when A is the Borel transform and B the Euler transform. *H. G. Eggleston.*

Amir (Jakimovski), Amnon. On a converse of Abel's theorem. Proc. Amer. Math. Soc. 3, 244–256 (1952).

The author considers the iteration of the Abel means A and the Cesàro means (C, α) [see the preceding review]. He calls a sequence s_n summable (A, α), $\alpha>-1$ to s if for its (C, α) means $\sigma_n^{(\alpha)}$, $(1-x)\sum\sigma_n^{(\alpha)}x^n\to s$ for $x\to 1-$. The main results are the inclusions $(A, \alpha)\supset(A, \beta)$, $\alpha\geqq\beta$ and $(A, \alpha)\supset(C, \beta)$, and the Tauberian theorems: if $\sigma_n^{(\alpha)}$, $\alpha>-1$ is slowly decreasing and A summable, then s_n is (C, α) summable; if s_n is (A, α) summable and $\sigma_n^{(\alpha)}$ is bounded, then s_n is $(C, \alpha+\epsilon)$ summable for each $\epsilon>0$. *G. G. Lorentz* (Kingston, Ont.).

Teghem, J. Sur des transformations de séries, à deux paramètres. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 37, 970–976 (1951).

Vermes [Acta Sci. Math. Szeged 14, 23–38 (1951); these Rev. 13, 27] has generalized the three one-parameter families of Taylor, Laurent, and Euler-Knopp series-to-series transformations by creating three corresponding two-parameter families of transformations in which the classical one-parameter families are embedded. For example, the Taylor-Vermes transform $A(\alpha, \alpha')$ of the series $\sum u_n$, defined by the equation

$$A(\alpha, \alpha') = \sum_{n=0}^{\infty}\sum_{\nu=n}^{\infty}\binom{\nu}{n}\alpha^{\nu-n}\alpha'^{n}u_\nu,$$

reduces to the classical Taylor transform $P(\alpha)$ of $\sum u_n$ in case $\alpha'=1-\alpha$. The author observes that the generalized transforms of the series $\sum u_n$ are always classical transforms of certain simply related series; for example, the transform $A(\alpha, \alpha')$ of $\sum u_n$ is the transform $P(\alpha/[\alpha+\alpha'])$ of the series $\sum u_n(\alpha+\alpha')^n$. This fact makes it possible to obtain properties of the two-parameter families from known properties of the classical one-parameter families. *G. Piranian.*

Jesmanowicz, L. On the Cesàro means. Studia Math. 12, 145–158 (1951).

Let $\{\sigma_n^\alpha\}$ denote the nth Cesàro means of order α of the series $\sum_{\nu=0}^{\infty}a_\nu$. The author proves the following results. (I) If $\sigma_n^\alpha=o(n^\beta)$, $\alpha+1\geqq 0$, $\alpha+\beta+1\geqq 0$, then the series $\sum a_\nu\nu^{-\beta-1}$ is summable $(C, \alpha-\epsilon)$ for $0<\epsilon<\alpha+1$. (II) Under the hypotheses in (I), $\sum a_\nu\nu^{-\beta}$ is either summable (C, α) or not summable by any Cesàro mean. (III) If $\sum a_\nu\nu^{-\beta}$ is summable (C, α) to zero, where $\beta+1>0$, $\alpha+1>0$, then $\sigma_n^\alpha=s+o(n^\beta)$, where $s=0$ for $\beta\geqq 0$. These are refinements on previously known results in which the hypotheses on α and β were more

restrictive [cf. J. M. Hyslop, Proc. Edinburgh Math. Soc. (2) 5, 182–201 (1938)]. *K. Chandrasekharan* (Bombay).

Čelidze, V. G. Cesàro summability of double numerical series. Soobščeniya Akad. Nauk Gruzin. SSR. 8, 121–126 (1947). (Russian)

Let $\sum a_{j,k}$ be a double series converging to s and having partial sums $s_{mn}=\sum_{j=0}^{m}\sum_{k=0}^{n}a_{j,k}$ such that (i) for each n, $s_{mn}/m\to 0$ as $m\to\infty$ and (ii) for each m, $s_{mn}/n\to 0$ as $n\to\infty$. Then the series is restrictedly evaluable $C_{1,1}$ to s. In this connection, see Čelidze [Doklady Akad. Nauk SSSR 60, 553–554 (1948); these Rev. 9, 507] and references given in the review. *R. P. Agnew* (Ithaca, N. Y.).

Čelidze, V. G. The mutual relation between Cesàro and Abel summation of double series. Soobščeniya Akad. Nauk Gruzin. SSR. 8, 365–372 (1947). (Russian)

It is proved that if $\sum a_{j,k}$ is a series whose $C_{1,1}$ transform $\sigma_{m,n}$ converges to s and if

$$\lim_{m\to\infty}\sigma_{m,n}/(m+1)^2=0,\quad \lim_{n\to\infty}\sigma_{m,n}/(n+1)^2=0,$$

then $\sum a_{j,k}$ is restrictedly evaluable to S by the Abel method. This result, and an application of it, were reviewed in greater detail [Čelidze, Doklady Akad. Nauk. SSSR 60, 553–554 (1948); these Rev. 9, 507] when the present paper was not available to Math. Rev. *R. P. Agnew.*

Sunouchi, Gen-ichiro. On a theorem of Hardy-Littlewood. Kōdai Math. Sem. Rep. 1951, 52–54 (1951).

The author gives proofs of the following two known results: (1) if $s_n=o(n^\beta)$ (C,p), $p>-1$, $\beta>0$, where $s_n=a_1+\cdots+a_n$, then $\sum n^{-\beta}a_n$ is summable (C,p) or not summable (C); (2) if $\sum n^{-\beta}a_n$ is summable (C,p), then $s_n=o(n^\beta)$ (C,p). [(1) and (2) were given for p an integer by Hardy and Littlewood, Proc. London Math. Soc. (2) 11, 411–478 (1913); (1) was given generally by Zygmund, Bull. Int. Acad. Polon. Sci. Cl. Sci. Math. Nat. Ser. A. Sci. Math. 1927, 309–331; (1) and (2) were given for $p\geqq 0$, with Riesz means, by Ananda Rau, Proc. London Math. Soc. (2) 34, 414–440 (1932) (Smith's Prize, Cambridge, 1918); the case $-1<p<0$ of (2) does not seem to be in the previous literature, but is contained in an independent result of the reviewer, ibid. (3) 1, 453–461 (1951); these Rev. 13, 548.] As a corollary of (1) the author completes a result given incorrectly by Jacob: (3) at a point of the Lebesgue set $\sum n^{-\delta}(a_n\cos nx+b_n\sin nx)$ is summable $(C,-\delta)$ [see Jacob, ibid. (2) 26, 470–492 (1927).]. *L. S. Bosanquet* (London).

Leslie, R. T., and Love, E. R. An extension of Mercer's theorem. Proc. Amer. Math. Soc. 3, 448–457 (1952).

A generalisation of Mercer's classical theorem on the arithmetical means of a real sequence $\{s_n\}$ to more general methods of summability. The main result is the following. Let (c_{nk}) be a convergence-preserving matrix such that $\lim_{n\to\infty}c_{nk}=c_k$, $\lim_{n\to\infty}\sum_{k=1}^{n}c_{nk}=c$, $\limsup_{n\to\infty}\sum|c_{nk}|=C$ exist. If $\{s_n\}$ is a bounded real sequence such that $s_n+q\sum_1^n c_{nk}s_k$ converges, then s_n converges provided that $|q|<|c-\sum_1^\infty c_k|/(C-\sum_1^\infty|c_k|)^2$ for all q when $\sum_1^\infty|c_k|=C$. In particular, if (c_{nk}) is a positive regular summation matrix, then $c_k=0$, $c=C=1$, so that $|q|<1$. This corollary covers the case of the arithmetical means, although in Mercer's theorem $\Re q>-1$ is only required and the s_n need not be bounded. *W. W. Rogosinski* (Newcastle-upon-Tyne).

Rajagopal, C. T. A note on generalized Tauberian theorems. Addendum. Proc. Amer. Math. Soc. 3, 457–458 (1952).

Details of proof of Lemma 9 in the author's paper in the same Proc. 2, 335–349 (1951); these Rev. 13, 28.

G. G. Lorentz (Kingston, Ont.).

de Bruijn, N. G., and Erdös, P. Some linear and some quadratic recursion formulas. I. Nederl. Akad. Wetensch. Proc. Ser. A. 54 = Indagationes Math. 13, 374–382 (1951).

de Bruijn, N. G., and Erdös, P. Some linear and some quadratic recursion formulas. II. Nederl. Akad. Wetensch. Proc. Ser. A. 55 = Indagationes Math. 14, 152–163 (1952).

Let $c_k>0$ and let $f(n)$ be defined by:

$$f(1)=1,\quad f(n)=\sum_1^{n-1}c_kf(n-k).$$

The program of this paper is the general study of the asymptotic behavior of the sequence $\{f(n)\}$ in terms of $\{c_k\}$. Let $C(x)=\sum_1^\infty c_kx^k$, and let $\gamma=\sup\{\alpha\,|\,C(\alpha)\leqq 1\}$. Because of the complexity of the results, it is possible only to cite representative theorems. Four subcases will be distinguished: case (i), $\gamma=0$, equivalent to $\limsup(c_n)^{1/n}=\infty$; case (ii), $\gamma>0$ and $C'(\gamma)<\infty$; case (iii), $C(\gamma)=1$ but $C'(\gamma)=\infty$; case (iv), $0<C(\gamma)<1$. In all cases, $\lim|f(n)|^{-1/n}=\gamma$, while in case (ii) $f(n)\sim\gamma^{-n}/C'(\gamma)$, and in case (iii) $f(n)=o(\gamma^{-n})$. The more complicated results deal with the behavior of the sequence $p_n=f(n)/f(n+1)$, and its relations to $\{c_n/f(n)\}$.

In case (ii) $\lim p_n$ exists, and in all cases, this limit when it exists is γ. If $\lim c_n/c_{n+1}=\gamma$, then $\lim p_n=\gamma$. The series $\sum c_n/f(n)$ is always divergent in case (i) and case (iv), and convergent in case (ii). If it converges in case (iii), then $\lim p_n=\gamma$. In case (i) (the study of which is reserved to the second of the papers), $\limsup c_n/f(n)=\infty$, and $\limsup c_n/f(n+1)=1$. Examples are given to show the wide variations of behavior that are possible here; in particular the lower limits of these may be zero. These results are also applied to the quadratic recursion relation: $f(1)=1$, $f(n)=\sum_1^{n-1}d_kf(k)f(n-k)$. The last section contains the following generalizations of a theorem of Pólya on subadditive functions [see Pólya and Szegö, Aufgaben und Lehrsätze . . . , Springer, Berlin, 1925, v. 1, p. 17]. Let $\phi(t)\uparrow$ and $\int\phi(t)t^{-2}dt<\infty$. Let $g(n)$ be approximately subadditive in the sense that $g(n+m)\leqq g(n)+g(m)+\phi(n+m)$ for all $n\geqq 1$, $m\geqq 1$ obeying $\frac{1}{2}\leqq m/n\leqq 2$. Then, $\lim g(n)/n$ exists (possibly $-\infty$). *R. C. Buck* (Madison, Wis.).

Miller, J. C. P. A method for the determination of converging factors, applied to the asymptotic expansions for the parabolic cylinder functions. Proc. Cambridge Philos. Soc. 48, 243–254 (1952).

The converging factor C_n of an asymptotic series

$$S(x)=u_0+u_1+\cdots+u_{n-1}\pm C_nu_n$$

may be expanded in a convenient way whenever $S(x)$ satisfies a linear differential equation so that C_n satisfies such an equation in x as well as a linear difference equation in n. If the number of changes of sign in u_n is infinite, C_n may be determined either from the differential equation or from the difference equation. If, however, the u_n's are ultimately all one-signed, both equations are needed. The method is outlined by its application to the asymptotic ex-

pansions for the solutions to Weber's equation $y''=(a+\frac{1}{4}x^2)y$. Numerical illustrations are given at the end of the paper.

H. A. Lauwerier (Amsterdam).

Vernotte, Pierre. Sur la sommation des séries asymptotiques de première espèce. C. R. Acad. Sci. Paris **234**, 1943–1945 (1952).

Agmon, Shmuel. A composition theorem for Dirichlet series. J. Analyse Math. **1**, 232–243 (1951). (English. Hebrew summary)

This is a composition theorem of the Hadamard type. Let D_f and D_g be two domains such that D_f contains a half-plane $\sigma>A_f$. Let S_f and S_g be their boundaries and let $S_{fg}=(\alpha+\beta\,|\,\alpha\varepsilon S_f,\ \beta\varepsilon S_g)$. The composition domain is by definition that component of the complement of the closure of S_{fg} which contains a right half-plane. Let $f(s)$, $s=\sigma+it$, be holomorphic in D_f, containing the half-plane $\sigma>A_f$, and let $\gamma(u)$ be positive and non-decreasing for $0\leqq u<\infty$. Then $f(s)$ is of growth $\gamma(u)$ in D_f if for any $\epsilon>0$ and any pair of real numbers $c_1<c_2$ there exists a constant $N=N(\epsilon; c_1, c_2)$ such that the maximum of $|f(\sigma+it)|$ is $\leqq N\gamma(|t|)$ for s in that part of D_f where $c_1\leqq\sigma\leqq c_2$ and s has a distance at least ϵ from S_f. Suppose now that $f(s)=\sum a_n e^{-\lambda_n s}$ and $g(s)=\sum b_n e^{-\mu_n s}$ where $\lambda_{n+1}-\lambda_n\geqq h>0$ and $|\lambda_n-\mu_n|\leqq k<h/2$ and let A_f and A_g be their abscissas of convergence. Suppose that $f(s)$ can be continued analytically into a domain D_f containing $\sigma>A_f$ and that $g(s)$ can be continued analytically into a domain D_g containing $\sigma>A_g$. Suppose that $f(s)$ is of growth $\gamma_f(u)$ in D_f and $g(s)$ of growth $\gamma_g(u)$ in D_g where $u^{-2}\log\gamma_f(u)$ and $u^{-2}\log\gamma_g(u)$ are in $L(1,\infty)$ and $\log\gamma_f(u)$ is also concave. If $h(s)=\sum a_n b_n e^{-\lambda_n s}$ then $h(s)$ converges absolutely for $\sigma>A_f+A_g$ and may be continued analytically into the composite region D_{fg}; furthermore, $h(s)$ is of growth $\gamma_f(u)$ in D_{fg}. Various extensions are indicated, in particular, to the case in which $f(s)$ has no other singularities than poles on the line of convergence.

E. Hille.

Skolem, Th. On the abscissa of convergence for some Dirichlet's series. Norske Vid. Selsk. Forh., Trondheim **24**, 54–59 (1951).

Let $f(x_1,\cdots,x_n)=\sum_{k=1}^{m}a_k x_1^{e_{k,1}}\cdots x_n^{e_{k,n}}$ where $a_k>0$, $e_{k,j}$ real, x_j positive integers, and the powers have their principal meaning. The author proves that the abscissa of convergence of the multiple series $\sum_{x_1=1}^{\infty}\cdots\sum_{x_n=1}^{\infty}[f(x_1,\cdots,x_n)]^{-s}$ is μ^{-1} where $\mu=\max_{t_1,\cdots,t_m}\min_k\sum_{j=1}^{n}e_{j,k}\delta_j$ with $\delta_j\geqq 0$, $\sum_1^n\delta_j=1$.

E. Hille (Paris).

Blambert, Maurice. Sur la composition des singularités des séries de Dirichlet générales. C. R. Acad. Sci. Paris **234**, 179–180 (1952).

Cette note complète une note antérieure concernant le même sujet [mêmes C. R. **230**, 1565–1567 (1950); ces Rev. **12**, 88]. L'auteur donne de nouvelles conditions pour qu'un point $\gamma=\alpha+\beta$, où α est un point singulier de $f(s)=\sum a_n e^{-\lambda_n s}$, et où β un point singulier de $\sum b_n e^{-\mu_n s}$, soit un point singulier de la fonction composée $\sum a_n^{(k)} b_n e^{-\mu_n s}$ (les $a_n^{(k)}$ sont formés à partir des a_n, λ_n, μ_n; k dépend du comportement de f et φ lorsque $t\to\infty$).

S. Mandelbrojt (Houston, Texas).

Tanaka, Chuji. Note on Dirichlet series. I. On the singularities of Dirichlet series. I. Tôhoku Math. J. (2) **3**, 285–291 (1951).

Let $\{c_n\}$ and $\{\lambda_n\}$ be two sequences, c_n real, $\lambda_n\geqq 0$, $\lambda_n<\lambda_{n+1}\uparrow\infty$. The sequence $\{c_n\}$ is said to have normal sign-changes (with respect to the sequence $\{\lambda_n\}$) if the sign-changes occur between $c_{n_{k-1}}$ and c_{n_k} and

$$\liminf_{k\to\infty}(\lambda_{n_k}-\lambda_{n_{k-1}})>0;$$

the sign-changes are normal with respect to a given sequence of nonoverlapping intervals $\{I_k\}$ if the subsequence $\{c_{n_j}\}$ whose corresponding exponents λ_{n_j} are located in the intervals $\{I_k\}$ has normal sign-changes. The author proves two theorems. I. If the Dirichlet series $\sum_1^\infty a_n e^{-\lambda_n s}$ is simply convergent for $\sigma>0$, then $s=0$ is a singular point provided there exist two sequences $\{x_k\}$, $\{\gamma_k\}$, $0<x_k\uparrow\infty$, γ_k real, such that

$$\text{(1)}\qquad \limsup_{k\to\infty}x_k^{-1}\log\Big|\sum_{[x_k]\leqq\lambda_n<x_k}\Re(a_n e^{-i\gamma_k})\Big|=0$$

where $[x]$ is the largest integer $\leqq x$, and (2) $\lim_{k\to\infty}\sigma_k/x_k=0$ where σ_k is the number of sign-changes of $\{\Re(a_n e^{-i\gamma_k})\}$ when λ_n is restricted to the interval $I_k=[[x_k](1-\omega),[x_k](1+\omega)]$, and (3) the sign-changes of $\{\Re(a_n e^{-i\gamma_k})\}$ are normal with respect to $\{I_k\}$. Here ω is a fixed number, $0<\omega<1$. Theorem II is of a similar nature: it is assumed that $x_k=\lambda_{n_k}$ and condition (1) is replaced by

$$\text{(1')}\qquad \limsup_{k\to\infty}\lambda_{n_k}^{-1}\log|\Re[a_{n_k}e^{-i\gamma_k}]|=0.$$

The proofs are based on the singularity criterion of A. Ostrowski [S.-B. Berlin. Math. Ges. **27**, 32–47 (1928)].

E. Hille (Paris).

Tanaka, Chuji. Note on Dirichlet series. II. On the singularities of Dirichlet series. II. Tôhoku Math. J. (2) **3**, 292–297 (1951).

The author gives a number of applications of the two theorems of the first note in this series [see the preceding review for details and notation]. Suppose that $F(s)=\sum_1^\infty a_n e^{-\lambda_n s}$ and $G(s)=\sum_1^\infty\Re(a_n)e^{-\lambda_n s}$ both have $\sigma_0=0$ as abscissa of simple convergence. If γ_k is taken identically zero, the convergence theorem of Kojima shows that it is always possible to find a sequence $\{x_k\}$ satisfying condition (1) of Theorem I. Conditions (2) and (3) show that $s=0$ is a singular point, in particular, if $\Re(a_n)\geqq 0$ for $[x_k](1-\omega)\leqq\lambda_n\leqq[x_k](1+\omega)$. Supposing $\gamma_k=0$ and that $\{x_k\}$ satisfies (1), $s=0$ will be singular for $F(s)$ if $\Re(a_n)\geqq 0$ for λ_n in I_k and $\lim[\cos\arg a_n]^{1/\lambda_n}=1$ when $\lambda_n\to\infty$ through the subsequence in the intervals I_k. The author also proves theorems of the Fabry type. Thus $\sigma=0$ is a natural boundary of $F(s)$ if there exists a subsequence $\{\lambda_{n_k}\}$ such that (a) $\limsup\lambda_{n_k}^{-1}\log|a_{n_k}|=0$, (b) $s_k/\lambda_{n_k}\to 0$ where s_k is the number of $a_n\neq 0$ corresponding to λ_n's in $I_k=[[\lambda_{n_k}](1-\omega),[\lambda_{n_k}](1+\omega)]$ and $\lim(\lambda_{n+1}-\lambda_n)>0$ when n tends to infinity in such a manner that λ_n and λ_{n+1} both belong to the same interval I_k. Here (b) may be replaced by (b') $\varphi_{n+1}-\varphi_n\to 0$ as $n\to\infty$ and λ_n, λ_{n+1} both are in I_k, where $\varphi_n=\arg a_n$. For various specializations these theorems give classical theorems of Landau, Pringsheim, Biggeri, Fekete, Carlson-Landau-Szász, and Fabry.

E. Hille (Paris).

Tanaka, Chuji. Note on Dirichlet series. VII. On the distribution of values of Dirichlet series on the vertical lines. Kôdai Math. Sem. Rep. **1952**, 5–8 (1952).

The author is concerned with the behavior of a function $F(s)=F(\sigma+it)$, defined by a Dirichlet series, on a vertical line in the half-plane of simple convergence. He proves that there are three and only three possibilities. (1) If $F(s)$ is bounded on the given line $\sigma=\sigma_1$ and also on some line $\sigma=\sigma_2$, $\sigma_0<\sigma_2<\sigma_1$, then $F(\sigma_1+it)$ is an almost periodic func-

tion of t. (2) If $F(s)$ is bounded on $\sigma=\sigma_1$ but on no line to the left of σ_1, the $F(\sigma_1+it)$ tends to no limit as $t\to+\infty$ or $-\infty$, and in any vertical strip $\sigma_1-\epsilon<\sigma<\sigma_1+\epsilon$, $F(s)$ assumes every finite value with at most one exception. (3) If $F(s)$ is unbounded on the line $\sigma=\sigma_1$, then it cannot happen simultaneously that

$$\lim_{t\to\infty}|F(\sigma+it)|=+\infty \quad \text{and} \quad |\arg F(\sigma_1+it)-\theta|\leqq\vartheta<\tfrac{1}{2}\pi$$

for fixed but arbitrary constants θ, ϑ with $0\leqq\theta<2\pi$.

E. Hille (Paris).

Fourier Series and Generalizations, Integral Transforms

Morse, Marston, and Transue, William. A new implication of the Young-Pollard convergence criteria for a Fourier Series. Duke Math. J. 18, 563–571 (1951).

A variety of criteria are known for the point-wise convergence of Fourier trigonometric series in one variable. Their relative strength has also been determined, more or less completely. Let φ be an even integrable function with period 2π. Then, sufficient conditions for the convergence (to zero) of its Fourier series at the origin are: either

$$(Y_P)\qquad \int_0^x \varphi(t)dt=o(x), \quad \int_0^x |d[t\varphi(t)]|=O(x),$$

or,

$$(L_Q)\qquad \int_x^\xi |\Delta_x(\varphi/t)|dt=o(1), \quad 0<x<\xi,$$

for some ξ with $0<\xi\leqq\pi$, where $\Delta_x F=F(t+x)-F(t)$. The authors here show that the Young-Pollard condition (Y_P) implies the Lebesgue-Gergen condition (L_Q). Hardy had previously shown that (L_Q) did not imply (Y_P). Since the authors apparently intended to label these criteria with the names of their discoverers, it is perhaps pertinent to mention an early paper by P. Noaillon [Acad. Roy. Belgique. Bull. Cl. Sci. (5) 3, 524–541 (1913)]. *K. Chandrasekharan.*

Izumi, Shin-ichi, and Sunouchi, Gen-ichirô. Notes on Fourier analysis. XLVIII. Uniform convergence of Fourier series. Tôhoku Math. J. (2) 3, 298–305 (1951).

The authors show that the condition

$$f(t)-f(u)=o(1/\log|t-u|^{-1})$$

as $t\to x$, $u\to x$, corresponding to the Dini-Lipschitz convergence test, also yields uniform convergence of the Fourier series of $f(t)$ at $t=x$. A similar statement prevails for the hypotheses of Young's convergence test. The conditions of the Hardy-Littlewood test, however, fail to yield uniform convergence. *P. Civin* (Eugene, Ore.).

Petersen, G. M. Means of Fourier constants. Trans. Roy. Soc. Canada. Sect. III. (3) 45, 33–38 (1951).

The author proves that if $f(x)\sim\sum a_n\cos nx$ belongs to one of the spaces $\Lambda(\alpha)$, $M(\alpha)$, $\Lambda(\alpha,p)$, $\Lambda(\alpha^*,p)$, recently introduced by Lorentz [Ann. of Math. (2) 51, 37–55 (1950); these Rev. 11, 442] and Lorentz and Halperin, then $\{A_n=n^{-1}\sum_{k=1}^n a_k\}$ is the sequence of Fourier coefficients of a function belonging to the same space as f. The same result holds for the sequence $\{B_n=\sum_{k=n}^\infty k^{-1}a_k\}$. The first part is an extension of a result of Hardy, proved for L^p spaces [Messenger of Math. 58, 50–52 (1928)], while the second part is an extension of a result of the reviewer [Bull. Amer. Math. Soc. 50, 741–744 (1944); these Rev. 6, 125] also for L^p spaces. *R. Bellman* (Stanford University, Calif.).

Boas, R. P., Jr. Integrability of trigonometric series. II. Math. Z. 55, 183–186 (1952).

Two theorems give sufficient conditions in order that a trigonometrical series (1) $\sum_{-\infty}^{\infty}c_n e^{inx}$ be a Fourier series. They are proved by using the Hausdorff-Young theorem and Hölder's inequality. Theorem I: If $1<q\leqq 2$, $q'=q/(q-1)$, $1\leqq p<q'$, $\alpha<1-q/p'$, then (1) is the Fourier series of a function $f(x)\in L^p$ if $c_n\to 0$ and $\sum_{n=-\infty}^{\infty}|c_{n+m}-c_{n-m}|^q=O(m^\alpha)$ as $m\to\infty$ through the multiples of some fixed integer. If $\alpha\geqq 1-q/p'$, the conclusion no longer holds. Theorem II: If $1<q\leqq 2$, $\beta<1$, $r\geqq q$, $r>1+\beta$, and if $c_n\to 0$ and $\sum_{n=-\infty}^{\infty}|c_{n+m}-c_{n-m}|^r=O(m^\beta)$ and in addition $\{c_n\}$ is a sequence of bounded variation, then (1) is the Fourier series of a function $f(x)\in L^p$ for $1\leqq p<(r-1)q/[r-1+(q-1)\beta]$.

B. Sz.-Nagy (Szeged).

***Vernotte, Pierre. Séries de Fourier, régularité, séries divergentes et formulation expérimentale.** Publ. Sci. Tech. Ministère de l'Air, no. 259, Paris, 1952. xviii+105 pp. 800 francs.

Following a preface of 16 pages on "La vérité nouvelle", the author gives discussions and calculations concerning approximation of functions by polynomials and trigonometric sums. *R. P. Agnew* (Ithaca, N. Y.).

Stečkin, S. B. On best approximations of periodic functions by trigonometric polynomials. Doklady Akad. Nauk SSSR (N.S.) 83, 651–654 (1952). (Russian)

Lozinskiĭ, S. M. The converse of Jackson's theorem. Doklady Akad. Nauk SSSR (N.S.) 83, 645–647 (1952). (Russian)

These two papers deal with the best approximation $E_n(f)$ to a given periodic function $f(x)$ by means of trigonometric polynomials. By $\omega_k(\delta,f)$ we denote the modulus of continuity of order k [cf. Stečkin, same Doklady (N.S.) 65, 135–137 (1949); Izvestiya Akad. Nauk SSSR. Ser. Mat. 15, 219–242 (1951); these Rev. 10, 529; 13, 29]. One of Stečkin's results was that $E_n(f)\asymp n^{-\alpha}$ if and only if $\omega_k(\delta,f)\asymp\delta^\alpha$ $(0<\alpha<k)$. He now proves that if $\alpha>0$, $0\leqq r<\alpha$, $k+r>\alpha$, then $E_n(f)\asymp n^{-\alpha}$ if and only if $\omega_k(\delta,f^{(r)})\asymp\delta^{\alpha-r}$.

Lozinskiĭ considers the same problem from a more general point of view. Let $\omega(u)$ be continuous, positive and non-decreasing in $(0,\infty)$, $\omega(0)=0$, and let it satisfy either (1) $\limsup_{u\to 0+}\omega(Ku)/\omega(u)<K^p$ with some $K>1$ or (2) $1<\liminf\omega(Ku)/\omega(u)\leqq\limsup\omega(Ku)/\omega(u)<K^p$ with some $K>1$. Lozinskiĭ then states the following results. If (1) holds, then $\omega_p(\delta,f)=O\{\omega(\delta)\}$ if and only if $E_n(f)=O\{\omega(n^{-1})\}$, $\omega_p(\delta,f)\asymp\omega(\delta)$ if and only if $E_n(f)\asymp\omega(n^{-1})$. If (2) holds, then the corresponding conditions connect $\omega_p(\delta,f^{(r)})=O\{\omega(\delta)\}$ with $E_n(f)=O\{n^{-r}\omega(n^{-1})\}$ and similarly with "$\asymp$." Finally, if there is no $K>1$ for which $\omega(u)$ satisfies (1) or (2), there is an f having the specified behavior of $E_n(f)$ or $E_n(f^{(r)})$ but not the specified behavior of the modulus of continuity.

R. P. Boas, Jr. (Evanston, Ill.).

Lozinskiĭ, S. M. On the convergence of interpolation processes for functions of two variables. Doklady Akad. Nauk SSSR (N.S.) 83, 789–792 (1952). (Russian)

The author states some sufficient conditions for the convergence of trigonometric interpolation polynomials in two dimensions. They are analogous to conditions given by him in an earlier paper for convergence of double Fourier series

[same Doklady (N.S.) **72**, 841–843 (1950); these Rev. **12**, 697], and are expressed in terms of concepts introduced there. *R. P. Boas, Jr.* (Evanston, Ill.).

Duffin, R. J., and Schaeffer, A. C. A class of nonharmonic Fourier series. Trans. Amer. Math. Soc. **72**, 341–366 (1952).

Let the sequence $\{\varphi_n\}$ of elements $\neq 0$ of a Hilbert space H constitute a frame in H, that is, let

$$A\|v\|^2 \leqq \sum |(v, \varphi_n)|^2 \leqq B\|v\|^2, \quad (0<A\leqq B \text{ fixed})$$

for all $v \varepsilon H$. The authors prove that any $v \varepsilon H$ may (perhaps in several ways) be expanded as $\sum b_n \varphi_n$, and also in terms of a conjugate frame as $\sum b_n' \varphi_n'$. The values β_n of b_n which make $\sum |b_n|^2$ a minimum form a moment sequence, $\beta_n = (u, \varphi_n)$. It is shown how the expansion coefficients β_n may be computed by a method of successive approximations if values of A and B are known. The removal of an element from a frame leaves either a frame or an incomplete set. If the latter is the case for every element the frame is called exact. For exact frames $\{\varphi_n\}$ and $\{\varphi_n'\}$ are biorthogonal. Exact frame, it turns out, is just another name for a sequence $\{\varphi_n\}$ which has a multiple $\{f_n\} = \{\lambda \varphi_n\}$, λ a constant, close to some complete orthonormal sequence $\{\psi_n\}$, close in the sense of Paley and Wiener: $\|\sum c_n(f_n - \psi_n)\|^2 \leqq \theta^2 \sum |c_n|^2$ (θ^2 fixed, <1) for all finite sequences of complex c_n.

A sequence $\{\lambda_n\}$ $(-\infty < n < \infty)$ of complex numbers is said to have uniform density (u.d.) d (>0) if $|\lambda_n - n/d| \leqq L$ for all n and $|\lambda_n - \lambda_m| \geqq \delta > 0$ $(n \neq m)$. It is shown that the sequence (*) $\{\exp(i\lambda_n t)\}$ $(-\infty < n < \infty)$ is a frame in $L_2(-\gamma, \gamma)$: (i) whenever $\{\lambda_n\}$ has u.d. $d > \gamma/\pi$; (ii) whenever $\{\exp(i\alpha_n t)\}$, $\alpha_n = \operatorname{Re} \lambda_n$, is a frame and $|\operatorname{Im} \beta_n| \leqq \beta$. By a result of Duffin and Eachus [Bull. Amer. Math. Soc. **48**, 850–855 (1942); these Rev. **4**, 97] (ii) implies that (*) is a frame in $L_2(-\pi, \pi)$ if $|\operatorname{Re} \lambda_n - n| \leqq a < \pi^{-1} \log 2$ and $|\operatorname{Im} \beta_n| \leqq \beta$. Since the transformation $f(z) = \int_{-\gamma}^{\gamma} g(t) e^{izt} dt$ defines a 1-1 correspondence between the $g(t) \varepsilon L_2(-\gamma, \gamma)$ and the $f(z) \varepsilon E(\gamma)$ (that is, the entire functions $f(z)$ which are $O(e^{\gamma|z|})$ and belong to $L_2(-\infty, \infty)$ along the real axis) the sequence (*) is a frame in $L_2(-\gamma, \gamma)$ if, and only if, there are constants $A' > 0$, $B' > 0$ such that

$$A' \leqq \sum |f(\lambda_n)|^2 \Big/ \int_{-\infty}^{\infty} |f(x)|^2 dx \leqq B'$$

for all $f(z) \varepsilon E(\gamma)$. Inequalities of this type (for example with $\lambda_n = n$) for the $f(z) \varepsilon E(\gamma)$ were first proved by Plancherel and Pólya [Comment. Math. Helv. **9**, 224–248 (1937); **10**, 110–163 (1937)]. Let $\{\lambda_n\}$ have u.d. 1, and let (*) be a frame in $L_2(-\pi, \pi)$. Then any $g(t) \varepsilon L_2(-\pi, \pi)$ may be expanded in $L_2(-\pi, \pi)$ as $\sum c_n \exp(i\lambda_n t)$. This (not necessarily unique) nonharmonic Fourier series of $g(t)$ on $(-\pi, \pi)$ associated with (*) is shown to have the property that

$$\Phi_N(t) = (\pi^2 - t^2) \sum_{-N}^{N} \{c_n \exp(i\lambda_n t) - a_n \exp(int)\} \to 0$$

uniformly on $-\pi \leqq t \leqq \pi$; here $\sum a_n \exp(int)$ denotes the ordinary Fourier series of $g(t)$ on $(-\pi, \pi)$. This result strengthens some results of Paley and Wiener [Fourier transforms in the complex domain, Amer. Math. Soc. Colloq. Publ., v. 19, New York, 1934; see theorem 38] and of N. Levinson [Gap and density theorems, Amer. Math. Soc. Colloq. Publ., v. 26, New York, 1940, see theorem 18; these Rev. **2**, 180]. *J. Korevaar* (Delft).

Mitchell, Josephine. On the spherical summability of multiple orthogonal series. Trans. Amer. Math. Soc. **71**, 136–151 (1951).

Following the analogy of the summability theory of simple orthogonal series on the one hand [cf. Kaczmarz, Studia Math. 1, 87–121 (1929); Kaczmarz and Steinhaus, Theorie der Orthogonalreihen, Warsaw-Lwów, 1935] and the theory of spherical summability of multiple Fourier series on the other [cf. Bochner, Trans. Amer. Math. Soc. **40**, 175–207 (1936)], the author extends, without difficulty, the main theorems on the convergence, absolute convergence and $(C, 1)$ summability of simple orthogonal series to the case of multiple orthogonal series summed over spheres. In particular, it is shown that summability $(C, 1)$ and summability (C, α), $\alpha > 0$, are equivalent in the latter case. Since all of these results are "exact generalizations both in statement and proof to the analogous theorems for simple orthogonal series", they are not restated here.

As can be expected, the author's results are proved for functions of class L_2. This fact is admitted as "a liability" by the author. It seems therefore that the author's comparison of Th. 3.3 with a result proved by the reviewer for functions of class L_1 is somewhat misleading. The statement (on p. 138) that in the case of spherical summability of multiple Fourier series "the behaviour of the function need only be known in the neighbourhood of the point considered" is again misleading, since S. Bochner [loc. cit] has established the existence of a critical exponent, such that for orders of Riesz summability not exceeding that exponent, summability is not a local property. The author's claim to novelty in having been able to study (C, α) summability for all $\alpha > 0$, is considerably diminished by the nature of the hypotheses imposed. If conditions on the "Fourier" coefficients are allowed to come in, even straight theorems on ordinary convergence can be established by Tauberian arguments [see, for instance, Theorems 4.1 and 6.1 in Chandrasekharan and Minakshisundaram, Duke Math. J. **14**, 731–753 (1947); these Rev. **9**, 279].

K. Chandrasekharan (Bombay).

***Nowacki, Werner. Fouriersynthese von Kristallen und ihre Anwendung in der Chemie.** Verlag Birkhäuser, Basel, 1952. 237 pp. Bound, 34.30 Swiss francs; unbound, 30.15 Swiss francs.

While this book is written for physicists and chemists who are interested in crystal structure, it is a convenient reference work for any mathematician who is interested in seeing how the formal theory of Fourier series is applied in this field. For example, the author tabulates the various forms which a three-dimensional Fourier series assumes for a function possessing one of the many possible kinds of symmetry. He also discusses the various methods which have been developed recently for attempting to discover the form of a Fourier series whose auto-convolution is given [unfortunately he does not cite the paper of Hartman and Wintner [Physical Rev. (2) **81**, 271–273 (1951); these Rev. **12**, 495] in which it is shown that the commonly accepted "facts" about the maxima of a function and its auto-convolution are incorrect; it is an unsolved problem to discover why the method works so well in practice]. One chapter is devoted to brief descriptions of computing devices of various types which are useful in Fourier synthesis.

R. P. Boas, Jr. (Evanston, Ill.).

Pistoia, Angelo. Sul prolungamento analitico della trasformata di Laplace. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **14(83)**, 481–492 (1950).

Es sei $f(p)=\int_0^\infty e^{-pt}F(t)dt$ und habe eine endliche Konvergenzabszisse. Der Verf. zeigt im Verallgemeinerung früherer Resultate von Amerio, dass

$$\lim_{n\to\infty}\int_0^\infty e^{-(p+\lambda_n i)t}\left(1+\frac{\lambda_n t}{n}\right)^n F(t)dt=f(p)$$

für gewisse Zahlfolgen λ_n auch noch in bestimmten Punkten links und auf der Konvergenzabszisse richtig ist, sofern sich $f(p)$ in diesen Punkten analytisch verhält. *W. Saxer.*

Tanaka, Chuji. Note on Laplace-transforms. I. On the determination of three convergence-abscissas. Mem. Fac. Sci. Eng. Waseda Univ. no. 15, 1–6 (1951).

Let $f(s)=\int_0^\infty \exp(-sx)\,d\alpha(x)$. Formulas are given for the abscissa of convergence σ_c, the abscissa of absolute convergence σ_a, and the abscissa of uniform convergence σ_u. For example $\sigma_u=\overline{\lim_{t\to\infty}}\,1/t\log T_t$ where

$$T_t=\sup_\tau\left|\int_{[t]}^t \exp(-i\tau x)d\alpha(x)\right|.$$

Here $[t]$ is the largest integer less than or equal to t. Again if $\phi(x)\epsilon\uparrow$, $\phi(x)/x\to\infty$, $\phi'(x)$ exists for $x>0$, and if for every $\epsilon>0$ the integral $\int_0^\infty \exp(-\epsilon x)\phi'(x)\,dx$ converges then $\sigma_u=\overline{\lim_{t\to\infty}}\,1/t\log T_t$ where

$$T_t=\sup_\tau\left|\int_0^t \exp(\phi(x)-\phi(t)-i\tau x)d\alpha(x)\right|.$$

I. I. Hirschman, Jr. (St. Louis, Mo.).

Tanaka, Chuji. Note on Laplace-transforms. II. On some class of Laplace-transforms. I. Kōdai Math. Sem. Rep. **1951**, 55–58 (1951).

Tanaka, Chuji. Note on Laplace-transforms. III. On some class of Laplace-transforms. II. Kōdai Math. Sem. Rep. **1951**, 59–60 (1951).

These papers contain known results on the coincidence of the abscissas of holomorphy, of convergence, of uniform convergence, and of absolute convergence of a Laplace transform when the object function is of exponential type in an angle. *I. I. Hirschman, Jr.* (St. Louis, Mo.).

Tanaka, Chuji. Note on Laplace-transforms. IV. On the determination of the regularity-abscissa. I. Kōdai Math. Sem. Rep. **1951**, 64–66 (1951).

Tanaka, Chuji. Note on Laplace-transforms. V. On the determination of the regularity-abscissa. II. Kōdai Math. Sem. Rep. **1951**, 67–70 (1951).

These papers contain a development for the Laplace transform of the Riesz summability process formulated originally for Dirichlet series. *I. I. Hirschman, Jr.*

Tanaka, Chuji. Note on Laplace-transforms. VI. On the distribution of zeros of partial sums of Laplace-transforms. Kōdai Math. Sem. Rep. **1951**, 96–99 (1951).

The author generalizes the theorems of Jentzsch and Szegö on the zeros of partial sums of power series to the Laplace transform. *I. I. Hirschman, Jr.*

Tanaka, Chuji. Note on Laplace-transforms. VII. On the overconvergence and singularities of Laplace-transforms. Kōdai Math. Sem. Rep. **1951**, 100–102 (1951).

The author generalizes the theorem of Ostrowski on overconvergent power series to the Laplace transform. This has been done previously by S. Rios [Revista Acad. Ci. Madrid **33**, 27–87 (1936)]. *I. I. Hirschman, Jr.*

Tanaka, Chuji. Note on Dirichlet series. VIII. On the singularities of Dirichlet series. V. Kōdai Math. Sem. Rep. **1952**, 9–12 (1952).

The author has found a necessary and sufficient condition in order that the point $s=\sigma_0+it_0$ on the line of convergence of a Laplace-Stieltjes integral be a singular point of the function so defined [see the third preceding review]. As an application of this theorem he proves a classical theorem, due to G. Pólya, on the existence of singular points on the line of convergence of a Dirichlet series [S.-B. Preuss. Akad. Wiss. **1923**, 45–50]. *E. Hille* (Paris).

Magnaradze, Leo. Abel's theorem for a double Laplace transform. Soobščeniya Akad. Nauk Gruzin. SSR. **8**, 113–119 (1947). (Russian)

Let the integral $\int_0^\infty\int_0^\infty f(x,y)dxdy$ exist [in a sense not stated] and have the value A. Let the function

$$s(x,y)=\int_0^x\int_0^y f(u,v)dudv$$

exist for $x,y>0$ and be such that (i) for each $y>0$, $s(x,y)/x\to 0$ as $x\to\infty$ and (ii) for each $x>0$, $s(x,y)/y\to 0$ as $y\to\infty$. Then the double Laplace transform

$$F(x,y)=\int_0^\infty\int_0^\infty e^{-xu-yv}f(u,v)dudv$$

converges restrictedly to A as $x,y\to 0+$; that is, if $0<\lambda<\Lambda$ then $F(x,y)\to A$ as $x,y\to 0+$ subject to the restriction $\lambda\leqq y/x\leqq\Lambda$. *R. P. Agnew* (Ithaca, N. Y.).

Opatowski, I. Laplace transform of $(\operatorname{erf}\sqrt{t})^2$. Amer. Math. Monthly **59**, 392 (1952).

***Ditkin, V. A., i Kuznecov, P. I. Spravočnik po operacionnomy isčisleniyu. Osnovy teorii i tablicy formul. [Handbook of operational calculus. Fundamentals of the theory and tables of formulas.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951. 255 pp. 7.30 rubles.

The first part (86 pp.) of this volume gives a summary (with proofs) of the more important properties of operational calculus. The material is organized in 12 Sections. 1. Introduction. 2. The Laplace integral and its principal properties. 3. Operators (based mainly on the Laplace-Carson integral) and the basic properties of the operators $D=d/dt$ and p. 4. Duhamel's integral (convolution). 5. Functions of D. 6. Some special operators (mostly algebraic functions of D, but $e^{-\lambda D}$ and $D^{1/2}e^{-\lambda D^{1/2}}$ are also treated). 7. Evaluation of operational expressions by contour integration (the complex inversion formula and some of its applications). 8. Composite operators of the form $u(D)\,F[q(D)]$. 9. Operational expressions depending on a parameter (this in preparation of the application of operational calculus to partial differential equations). 10. Applications to differential equations. 11. Boundary value problems for the heat equation and wave equation (both generalized). 12. Some problems in heat conduction. A brief bibliography (29 items) is added.

The second part (159 pp.) contains an extensive table of transform pairs $f(t)$ and $F(p)=p\int_0^\infty f(t)e^{-pt}dt$ arranged according to $F(p)$. There is a list (8 pp.) of notations and definitions of higher functions followed by sixteen sections.

O. General formulas (55). I. Rational functions (163). II. Irrational functions and powers with arbitrary indices (171). III. Exponential functions (212). IV. Trigonometric and hyperbolic functions (89). V. Logarithmic, inverse trigonometric, and inverse hyperbolic functions (128). VI. Gamma and related functions (57). VII. Exponential integral and related functions (50). VIII. Confluent hypergeometric functions and particular cases (90). IX. Bessel and related functions (179). X. Legendre functions (15). XI. Elliptic and similar integrals (25). XII. Theta functions (15). XIII. Mathieu functions (12). XIV. Hypergeometric series (55). XV. Miscellaneous functions (38).

A. Erdélyi (Pasadena, Calif.).

Polynomials, Polynomial Approximations

Ilieff, Ljubomir. Über die Newtonschen Näherungswerte. Annuaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1. **46**, 167–171 (1950). (Bulgarian. German summary)

Let $F(z)$ be a polynomial. If the Newton sequence $z_n=z_{n-1}-F(z_{n-1})/F'(z_{n-1})$, $n=1, 2, \cdots$, converges it converges to a root α_k of $F(z)$. Tschakaloff has suggested the problem of finding the domains $E(\alpha_k)$ in which z_0 must lie in order that the sequence converges to α_k. The author considers the special case in which all the roots are real and simple and proves that a certain interval I_k on the real axis is in $E(\alpha_k)$, $k=1, 2, \cdots, n$. *A. W. Goodman.*

Pólya, G. Remarques sur un problème d'algèbre étudié par Laguerre. J. Math. Pures Appl. (9) **31**, 37–47 (1952).

A theorem due to Laguerre [Oeuvres, tome 1, Gauthier-Villars, Paris, 1898, pp. 119–122] states that if

$$f(z)=(z-x_1)(z-x_2)\cdots(z-x_n)=z^n+a_1z^{n-1}+\cdots+a_n,$$

then the elementary symmetric functions of the roots $(-1)^ka_k$ are rational functions of $s_1, s_2, \cdots, s_{2n-1}$, where $s_k=x_1^k+x_2^k+\cdots+x_n^k$. The author adjoins to $f(z)$ a second polynomial

$$g(z)=(z-y_1)(z-y_2)\cdots(z-y_n)=z^n+b_1z^{n-1}+\cdots+b_n,$$

and proves that the quantities R, Ra_j, Rb_j $(j=1, 2, \cdots, n)$ are polynomials in $u_1, u_2, \cdots, u_{2n}$ where $R=\prod_{i,j=1}^{n}(x_i-y_j)$ and $u_k=\sum_{i=1}^{n}(x_i^k-y_i^k)$. The theorem is best possible in the sense that $R=R(u_j)$ is irreducible. This generalization of the Laguerre result yields it as a special case, when $y_j=-x_j$ $(j=1, 2, \cdots, n)$. In this case the denominator R, in the rational expressions for the a_j, is now reducible. This permits the author to sharpen Laguerre's result, namely if $P=\prod_{i<j}^{n}(x_i+x_j)$, then P, $Pa_1, \cdots, Pa_n$, are polynomials in $s_1, s_2, \cdots, s_{2n-1}$, and $P=P(s_j)$ is irreducible.

A. W. Goodman (Lexington, Ky.).

Džrbašyan, M. M. On the growth of derivatives of polynomials. Doklady Akad. Nauk SSSR (N.S.) **84**, 5–8 (1952). (Russian)

The author proves the following theorem. Let $p(x)$ be even, differentiable, and satisfy $xp'(x)\uparrow\infty$ as $x\uparrow\infty$, and let $g(y)$ be its inverse, supposed increasing and not zero for $y\geqq 1$. Then there exist functions $A_1(R)$, $A_2(R)$ of R alone such that, for a polynomial $Q_n(x)$ of degree n satisfying $|Q_n(x)|\leqq\exp\{p(x)\}$ on the whole real axis, we have on $(-R, R)$

$$|Q_n'(x)|\leqq A_1(R) \quad \text{if} \quad \int^{\infty} x^{-2}p(x)dx<\infty,$$

$$|Q_n'(x)|\leqq A_2(R)\int_1^n\{g(y)\}^{-1}dy \quad \text{if} \quad \int^{\infty} x^{-2}p(x)dx=\infty.$$

This implies in particular a result of S. Bernšteĭn on polynomial approximation [same Doklady (N.S.) **77**, 549–552 (1951); these Rev. **12**, 814]. *R. P. Boas, Jr.*

Huff, William N., and Rainville, Earl D. On the Sheffer A-type of polynomials generated by $\phi(t)f(xt)$. Proc. Amer. Math. Soc. **3**, 296–299 (1952).

Huff [Duke Math. J. **14**, 1091–1104 (1947); these Rev. **9**, 282] obtained a necessary and sufficient condition that a polynomial set $\{y_n(x)\}$ with generating function of the form $\phi(t)f(xt)=\sum_0^{\infty}y_n(x)t^n$ be a set of A-type k. The present note obtains an equivalent condition, namely that $f(x)$ be a hypergeometric function with k denominator parameters and no numerator parameters: $f(x)={}_0F_k(-;\beta_1, -m, \beta_k; \sigma x)$, with σ a non-zero constant. *I. M. Sheffer.*

Kössler, Miloš. Simple polynomials. Czechoslovak Math. J. **1(76)**, 5–15 (1951) = Čehoslovack. Mat. Ž. **1(76)**, 5–17 (1951).

For a polynomial $P(z)=\sum_{k=1}^{n}a_kz^k$, $a_1=1$, $|a_n|>0$, consider the associated system:

$$1+\sum_{1}^{n}a_kx^{k-1}P_{k-1}(u)=0, \quad x^{n-1}+\sum_{1}^{n}\bar{a}_kx^{n-k}P_{k-1}(u)=0,$$

where

$$P_{k-1}(u)=\sum_{r=0}^{r\leq\frac{1}{2}(k-1)}(-1)^r\binom{k-r-1}{r}u^{k-2r-1}.$$

Let $R(u)=\sum_{k=0}^{N}A_ku^{N-k}$, $N\leqq 2(n-1)^2$ be the resultant of the associated system. It is proved that $P(z)$ is schlicht in $|z|<r, r>1$, if and only if $R(u)$ does not vanish identically and has no real root lying in $-2\leqq u\leqq 2$. Detailed discussion of the case $n=3$ follows. *P. Davis* (Washington, D. C.).

Castoldi, Luigi. Un particolare sistema ortonormale per lo sviluppo in serie di una funzione assegnata su un intervallo infinito. Atti Accad. Ligure **7**, 311–315 (1951).

Let $f(x)$ be continuous together with its first derivative on $-\infty<x<\infty$ with the exception of a finite number of discontinuities of the first kind. Let $f(x)$ be normalized at these points. Suppose further that $f(x)$ approaches a (finite) limit as $x\to-\infty$ and $x\to+\infty$, and that $f'(x)=O(1/x^2)$ as $|x|\to\infty$. Under these conditions $f(x)$ is shown to have a convergent expansion with respect to a particular orthogonal set on $(-\infty, \infty)$ with weight function $(1+x^2)^{-1}$.

Set $x=\tan\varphi$, $F(\varphi)=f(\tan\varphi)$; then F is defined on $-\pi/2<\varphi<\pi/2$, and F, F' are piece-wise continuous on this interval. Extend the definition of F to $-3\pi/2<\varphi\leqq-\pi/2$ by the relations $F(-3\pi/2+h)=-F(\pi/2-h)$ for $0<h<\pi$, and $F(-\pi/2)=0$; then $T(\theta)=F(\varphi)$ (where $\theta=\varphi+\pi/2$) is an odd function defined on $-\pi<\theta<\pi$, and has there a convergent Fourier sine development

$$T(\theta)=\sum_{n=1}^{\infty}b_n\sin n\theta$$

$$=\sum_{m=0}^{\infty}(-1)^m[b_{2m}\sin 2m\varphi+b_{2m+1}\cos(2m+1)\varphi].$$

Returning to x, set $\cos\varphi=(1+x^2)^{-1/2}$, $\sin\varphi=x(1+x^2)^{-1/2}$, and define $X_{2m+1}(x)=\cos(2m+1)\varphi$, $Y_{2m}(x)=\sin 2m\varphi$. The function $(1+x^2)^{(m+1)}X_{2m+1}(x)$ is a polynomial in x of degree $2m$, and $(1+x^2)^m Y_{2m}(x)$ of degree $2m-1$. The above Fourier series for $T(\theta)$ becomes the convergent series $f(x)=\sum_{m=0}^{\infty}[b_{2m}Y_{2m}(x)+b_{2m+1}X_{2m+1}(x)]$, with

$$b_{2m}=\frac{2}{\pi}\int_{-\infty}^{\infty}f(\xi)\frac{Y_{2m}(\xi)}{1+\xi^2}d\xi,\quad b_{2m+1}=\frac{2}{\pi}\int_{-\infty}^{\infty}f(\xi)\frac{X_{2m+1}(\xi)}{1+\xi^2}d\xi.$$

And the system $X_{2m+1}(x)$, $Y_{2m}(x)$ is an orthogonal set with weight function $(1+x^2)^{-1}$. *I. M. Sheffer.*

*Bernštein, S. N. **On the best approximation of functions of several variables by means of polynomials or trigonometric sums.** Trudy Mat. Inst. Steklov., v. 38, pp. 24–29. Izdat. Akad. Nauk SSSR, Moscow, 1951. (Russian) 20 rubles.

The author studies the best approximation in a rectangle to a continuous function $F(x, y)$ of two variables by reducing the problem to that of approximating by functions of the form $\sum_{k=0}^{m}a_k(x)y^k$, where $a_k(x)$ are continuous (or the analogous sums with x and y interchanged). Denote the best approximation to $F(x, y)$ by polynomials of degree n in x and m in y by $E_{n,m}F(x, y)$; the best approximation by polynomials of degree m in y with degree in x unrestricted, by $E_mF((x), y)$; and the maximum of the latter quantity for varying x, by $E_{\infty,m}F(x, y)$. The author obtains inequalities like

$$E_{n,m}F(x, y)\leqq(2\pi^{-1}+\delta_m)[E_{n,\infty}F(x, y)+E_{\infty,m}F(x, y)]\log m,$$

where $\delta_m\to 0$, or the same with δ_m and $\log m$ replaced by δ_n and $\log n$. He gives more precise results which cannot be briefly summarized and concludes with an application to trigonometric approximation. *R. P. Boas, Jr.*

Davis, Philip, and Pollak, Henry. On an equivalent definition of the transfinite diameter. Proc. Amer. Math. Soc. **3**, 152–155 (1952).

Soit B un domaine simplement connexe limité par une courbe de Jordan, a son diamètre transfini; posons $\lambda_n^p=\min(\iint_B|Q_n(z)|^p\,dx\,dy)^{1/p}$ parmi tous les polynomes $Q_n(z)=z^n+\cdots$; alors on a $\lim(\lambda_n^p)^{1/n}=a$ pour $p\geqq 2$. *J. Favard* (Paris).

Special Functions

Chakravarty, Nalini Kanta. On some operational and other relations involving Tschebyscheff's and Laguerre polynomials. Bull. Calcutta Math. Soc. **43**, 71–76 (1951).

A sample of the relations derived in this paper is

$$\frac{(2r)!}{2\sqrt{\pi}}\sum_{s=0}^{r}\frac{\Gamma(r-s-\frac{1}{2})}{2^{2s}(r-s)!(2s)!}[H_{2s}(\sqrt{\tfrac{1}{2}}x)]^2=2rL_{2r-1}(x)-L_{2r}(x).$$

A. Erdélyi (Pasadena, Calif.).

Deverall, L. I., and Thorne, C. J. Some relations involving special functions. Math. Mag. **25**, 183–188 (1952).

Starting with the formula for the sum of a finite geometric series and using a short table of Laplace transforms the authors derive eight identities among special functions. The special functions that are involved here include gamma and Bessel functions and Laguerre and Hermite polynomials. The following is a sample of those identities:

$$\frac{2^n t^n}{1\cdot 3\cdot 5\cdots(2n-1)}=1+\sum_{i=1}^{n}\binom{n}{i}\frac{i!}{(2i)!}H_{2i}(\sqrt{t}),$$

where $H_n(t)$ is the Hermite polynomial. [Reviewer's note. The lower limit of the integral in the author's formula (21) should be k rather than 0; with that correction the identity reduces to one that is independent of k. The author's identity (38) is incorrect because it is based upon an erroneous transform of the function $t^n\sin at$ printed on page 296 of the reviewer's Modern operational mathematics in engineering [McGraw-Hill, New York, 1944; these Rev. **5**, 267].] *R. V. Churchill* (Ann Arbor, Mich.).

Szász, Otto. On the relative extrema of the Hermite orthogonal functions. J. Indian Math. Soc. (N.S.) **15** (1951), 129–134 (1952).

Denote by $\mu_{0,n}, \mu_{1,n}, \cdots$ the successive relative maxima of $|\phi_n(x)|$ as x decreases from $+\infty$ to 0, where the $\phi_n(x)$ are the Hermite orthogonal functions. It is shown that $\mu_{r,n}>\mu_{r,n+1}$ for $n\geqq r\geqq 0$. Corresponding results for the other classical systems have already been established [Szegő, Boll. Un. Mat. Ital. (3) **5**, 120–121 (1950); J. Todd, ibid., 122–125; O. Szász, ibid., 125–127; these Rev. **12**, 332]. Combining this result with the classical fact that $\mu_{r,n}>\mu_{r+1,n}$, $r=0, 1, \cdots$, it follows that $|\phi_n(x)|\leqq 1$, confirming a conjecture of G. E. H. Reuter [J. London Math. Soc. **24**, 159–160 (1949); these Rev. **11**, 29]. *J. Todd.*

Bagchi, Hari Das, and Chatterjee, Phatik Chand. Linear difference equations associated with certain special functions. J. Indian Math. Soc. (N.S.) **16**, 37–40 (1952).

It is well known that every sequence of functions $\{f_n(z)\}$ which obeys the recurrence relation $z(f_{n-1}+f_{n+1})=2f_n$ and the differentiation formula $f_{n-1}-f_{n+1}=2f_n'$, also satisfies Bessel's differential equation. In this paper the corresponding work is carried out for Neumann's polynomials O_n and Schläfli's polynomials S_n, both of which satisfy inhomogeneous Bessel equations. *A. Erdélyi* (Pasadena, Calif.).

Gatteschi, Luigi. Valutazione dell'errore nella formula di Mc Mahon per gli zeri della funzione

$$J_0(kz)Y_0(z)-J_0(z)y_0(kz).$$

Ann. Mat. Pura Appl. (4) **32**, 271–279 (1951).

The author derives, by a new method, McMahon's expansion for the higher roots of the equation

$$J_0(kz)Y_0(z)-J_0(z)Y_0(kz)=0$$

in z. His method enables him to estimate the error rather accurately, and he determines, for instance, for $k=1.5$, the fifth root of the equation to within 5 units of the 8th decimal (10th significant figure). *A. Erdélyi* (Pasadena, Calif.).

van Wijngaarden, A. On a certain asymptotic expansion. Computation Dept., Math. Centre, Amsterdam, Rep. R 177, i+3 pp. (1952).

The asymptotic expansion of

$$f(z)=\sum_{k=1}^{\infty}k^{-1/2}(k!)^{-2}(z/2)^{2k}$$

for large z is obtained by expanding $(k!\sqrt{k})^{-1}$ in an inverse factorial series. Interchange of the order of summations gives a convergent expansion of $f(z)$ whose terms contain modified Bessel functions of variable z; and asymptotic ex-

pansions of these Bessel functions lead to the required asymptotic expansion of $f(z)$. 6 terms of the series reproduce $f(20)$ with an error of one unit in the sixth significant figure.
A. Erdélyi (Pasadena, Calif.).

Agostinelli, Cataldo. Sulle funzioni epicicloidali e alcune nuove relazioni fra le funzioni di Bessel. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **11**, 339–344 (1951).

The author develops further the solutions in epicycloidal coordinates of the wave equation [see same Rend. (8) **7**, 316–320 (1950); these Rev. **11**, 440] and compares these solutions with standard solutions in polar coordinates, thus obtaining certain relations involving Bessel functions.
A. Erdélyi (Pasadena, Calif.).

Chandra, Dinesh. On the Hankel transformation of generalized hypergeometric functions. J. Indian Math. Soc. (N.S.) **16**, 41–45 (1952).

The author evaluates the Hankel transform of order ν of

$$x^{3\rho_s-\nu-3/2}\,{}_qF_s(\alpha_r;\rho_s;-\tfrac{1}{4}x^2)$$

as a linear combination of ${}_sF_s$ series. [Note that the variable $-\frac{1}{4}x^2$ is misplaced in the statement of the result on p. 41; it should be at the end of line 5 up.] *A. Erdélyi.*

Roothaan, C. C. J. A study of two-center integrals useful in calculations on molecular structure. I. J. Chem. Phys. **19**, 1445–1458 (1951).

The author develops formulas for the computation of integrals of the forms

$$\int \chi_a M \chi_b dv, \quad \int\int \chi_a(1)\chi_b(2)(r_{12})^{-1}\chi_a'(1)\chi_b'(2)dv_1dv_2$$

where a and b indicate the centres to which the "Slater type atomic orbitals" are referred. The χ expressed in spherical polars are surface harmonics multiplied by a power of the radial distance times an exponential function of this distance. M is usually a simple function of the radial distances. Explicit formulas are given for a large number of integrals, and reduction formulas for others. The bibliography gives references to other papers in which evaluations of such integrals are found and lists the integrals evaluated in each of the papers referred to. *A. Erdélyi.*

Rüdenberg, Klaus. A study of two-center integrals useful in calculations on molecular structure. II. The two-center exchange integrals. J. Chem. Phys. **19**, 1459–1477 (1951).

The integrals discussed in this paper are of the form

$$\int\int \chi_a(1)\chi_b(1)(r_{12})^{-1}\chi_c(2)\chi_d(2)dv_1dv_2$$

where the χ are as in the preceding review except that they are now referred to four different centres $a, \cdots, d$. Products such as $\chi_a\chi_b$ are represented in prolate spheroidal coordinates (with a and b as foci), and for 65 distributions the coefficients of this representation are tabulated. The evaluation then proceeds in terms of certain auxiliary functions.
A. Erdélyi (Pasadena, Calif.).

Lah, Ivo. Eine neue Funktion der Versicherungsmathematik und ihre Anwendung. Mitt. Verein. Schweiz. Versich.-Math. **51**, 191–210 (1951).

The new function is closely related to repeated sums of the D_x table and so to increasing annuities. The function is used in expressing the variation of the annuity with varying interest in generalized formulae of the Poukka type.
P. Johansen (Copenhagen).

Differential Equations

Bulgakov, B. V. On equivalence and consistency of systems of linear differential equations with constant coefficients. Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 15–22 (1952). (Russian)

The author corrects certain results by N. N. Luzin [Avtomatika i Telemehanika **1940**, no. 5, 3–66; these Rev. **8**, 273]. Consider a system (*) $f(D)y=w(t)$, where $f(D)$ is a matrix whose elements are polynomials in $D=d/dt$, $w(t)$ a column vector. Any $i\times i$ minor of the "augmented" matrix $(f(D), w(t))$ is either a minor of $f(D)$ or else is formed with i rows, $(i-1)$ columns of $f(D)$ and the column w; in the latter case, the "reduced" minor is defined thus: develop the minor as a sum of products of elements of w by $(i-1)\times(i-1)$ minors of $f(D)$, divide these by the M.C.D. of all the $(i-1)\times(i-1)$ minors formed with the $(i-1)$ columns concerned and then perform the D-operation. The "rank" of the augmented matrix is defined as the maximum order of the minors of $f(D)$ and of the reduced minors which do not vanish identically. The main result is: In order that (*) be consistent it is necessary and sufficient that $f(D)$ and the augmented matrix have the same "rank".
J. L. Massera (Montevideo).

Sansone, Giovanni. Equazioni differenziali nel campo reale: comportamento asintotico delle soluzioni; punti singolari; soluzioni periodiche e valutazione del periodo. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **10**, 265–289 (1951).

This is an expository article which summarizes, in a clear and convenient way, many known results concerning solutions of ordinary differential equations in the real domain. Among the topics discussed are the asymptotic behaviors of solutions of linear equations, the properties of solutions in the neighborhoods of singular points, and the existence and properties of periodic solutions. *L. A. MacColl.*

***Péyovitch, T. Sur les solutions asymptotiques des équations différentielles.** Société des Mathématiciens et Physiciens de la R. P. de Serbie, Éditions spéciales. Naučna Knjiga, Belgrade, 1952. 52 pp.

The solutions of the system

$$(1)\qquad \frac{dx_i}{dt}+\sum_{k=1}^{n}\bar{a}_{ik}x_k=f_i(t)+F_i(t,x_1,x_2,\cdots,x_n)$$

are studied with respect to their behavior as $t\to+\infty$. Here, the $\bar{a}_{ik}$ are constants, and the continuous functions $f_i(t)$ satisfy $\lim_{t\to\infty} f_i(t)=0$. The $F_i(t,x_1,\cdots,x_n)$ are in a certain domain continuous and satisfy a Lipschitz condition. Moreover, $F_i(t,0,0,0)=0$. The special case $F_i\equiv 0$ is treated first, and it is shown that if no characteristic exponent has real part zero, there exist solutions $x_i(t)$ such that $\lim_{t\to\infty} x_i(t)=0$. These solutions involve as many arbitrary constants as there are characteristic exponents with negative real part. If characteristic exponents with real part zero are present the statement subsists, provided certain integrals involving the data converge. By the method of successive approximation this result is then extended to the case of general F_i.

[As far as the reviewer has been able to make out from the not very clear exposition the author uses in an essential manner an additional, unstated, assumption limiting the size of the constant in the Lipschitz condition.]

W. R. Wasow (Los Angeles, Calif.).

de Castro Brzezicki, A. On the asymptotic behavior of integrals of linear differential equations. Revista Acad. Ci. Madrid **45**, 351–362 (1951). (Spanish)

The author gives several theorems concerning the asymptotic behavior of the solutions of linear differential equations. Both the methods and the results closely resemble those which are given in G. Sansone's "Equazioni differenziali nel campo reale" [Zanichelli, Bologna, 1949; these Rev. **11**, 32], Parte seconda, pp. 15–24. The author asserts that his proofs are simpler, and his results more precise, than those given by Sansone. However, it is not clear to the reviewer that this assertion is justified. *L. A. MacColl.*

Reuter, G. E. H. Boundedness theorems for non-linear differential equations of the second order. II. J. London Math. Soc. **27**, 48–58 (1952).

The equation (*) $\ddot{x}+kF(\dot{x})+g(x)=kp(t)$ is said to be of class B if there exist constants B_1 and B_2 independent of k, $0<k\leqq 1$, such that for any solution of (*) $|x(t)|<B_1$, $|\dot{x}(t)|<B_2$ for $t>t_0$ (where t_0 depends on the particular solution). Let g, F and p be continuous functions and let $g(x)\operatorname{sgn} x\to\infty$ as $|x|\to\infty$, $F|y|\operatorname{sgn} y\to\infty$ as $|y|\to\infty$, and $p(t)$ be bounded. Then (*) is of class B. The equation (†) $\ddot{x}+g(x)=k\psi(x,\dot{x},t)$ is of class B if $g(x)$ is as above, ψ is continuous, and $\psi(x,y,t)\operatorname{sgn} y<A_1+A_2|y|$ for all x, y, t and $\psi\operatorname{sgn} y<-A_3|y|$ for $|x|\geqq a_1$, $|y|\geqq b_1$ and all t where A_1, A_2, A_3, a_1, b_1 are positive constants independent of k. *N. Levinson* (Cambridge, Mass.).

Minorsky, Nicolas. Sur l'interaction des oscillations non linéaires. C. R. Acad. Sci. Paris **234**, 292–294 (1952).

The differential equation

$$\ddot{x}+\epsilon(x^2-1)\dot{x}+[1+(a-cx^2)\cos 2t]x=0,$$

in which ϵ, a and c are small parameters of the same order of magnitude, can be regarded as a generalization both of Van der Pol's and of Mathieu's equation. By means of a method developed in some of his previous papers [see, e.g., same C. R. **232**, 2179–2180 (1951); these Rev. **13**, 38] the author discusses the stable periodic solutions of this differential equation. For certain ranges of the parameters two stable oscillations are possible giving rise to the phenomenon of oscillation hysteresis. *W. Wasow.*

Fränz, Kurt. Theory of high precision linear servomechanisms. Revista Unión Mat. Argentina **15**, 128–148 (1952). (Spanish)

An expository article dealing, without essential novelty, with the errors of stable linear servomechanisms, and with the design of servomechanisms of high precision.

L. A. MacColl (New York, N. Y.).

Campbell, J. G. A criterion for the polynomial solutions of a certain Riccati equation. Amer. Math. Monthly **59**, 388–389 (1952).

Titchmarsh, E. C. Some theorems on perturbation theory. III. Proc. Roy. Soc. London. Ser. A. **207**, 321–328 (1951).

With a certain boundary condition at $x=0$, the spectrum of the differential equation

$$\frac{d^2\varphi}{dx^2}+\lambda\varphi=0 \tag{1}$$

is continuous over $0<\lambda<\infty$, and has an isolated negative eigenwert, whereas the spectrum of the perturbed equation

$$\frac{d^2\varphi}{dx^2}+\{\lambda-\epsilon x\}\varphi=0 \tag{2}$$

is continuous over $-\infty<\lambda<\infty$. The matter at issue is the way in which the spectrum of (1) is approached by that of (2) when $\epsilon\to 0$ through positive values. It is concluded that the spectrum of (2) is "highly concentrated" in the neighborhood of a point which corresponds in a specific way with the isolated eigenwert of (1). *R. E. Langer.*

Titchmarsh, E. C. Some theorems on perturbation theory. IV. Proc. Roy. Soc. London. Ser. A. **210**, 30–47 (1951).

In the paper reviewed above the author considered a case in which the spectrum of the differential equation

$$\frac{d^2\varphi}{dx^2}+\{\lambda-q(x)\}\varphi=0$$

is discrete over the range $-\infty<\lambda<0$, whereas that of the perturbed equation

$$\frac{d^2\varphi}{dx^2}+\{\lambda-q(x)-\epsilon\sigma(x)\}\varphi=0$$

is continuous. The case was that of $q(x)\equiv 0$, $\sigma(x)=-x$. It was shown that the formula which gives the nth perturbed eigenvalue approximately, when it exists, gave approximately a point in the neighborhood of which the perturbed spectrum is highly concentrated. The present paper considers the problem for functions $q(x)$ and $\sigma(x)$ which are not specific, though they are subject to quite restrictive hypotheses and specifications. *R. E. Langer.*

Shah, S. M. Note on eigenfunction expansions. J. London Math. Soc. **27**, 58–64 (1952).

The paper is concerned with the differential equation

$$y''+(\lambda+x^2)y=0,$$

with a boundary condition at $x=0$, and the condition that the solution be $L^2(0,\infty)$. The spectrum of this is known to be continuous, and a formula analogous to Fourier's integral is also known. By an evaluation of residues the author derives series expansions for a function $F(z)$. These are in terms of Weber functions and Hermite polynomials (the differential equation is a form of Weber's equation). Various conditions of analyticity, etc. are imposed on $F(z)$.

R. E. Langer (Madison, Wis.).

Levitan, B. M. On a theorem of H. Weyl. Doklady Akad. Nauk SSSR (N.S.) **82**, 673–676 (1952). (Russian)

Assuming the eigen-function expansion for

$$y''+(\lambda-q(x)y=0$$

over $(0,\infty)$, where $q(x)$ is summable over every finite interval, the author shows how to construct for non-real λ a solution which is of $L^2(0,\infty)$. The existence of such a solution was found by H. Weyl [Math. Ann. **68**, 220–269 (1910)]. The present method has interest in that the author

has given previously [Doklady Akad. Nauk SSSR (N.S.) 73, 651–654 (1950); these Rev. 12, 502] a method of proving the expansion theorem without using Weyl's result. He also gives an expression for the required L^2 solution in terms of a fundamental pair $\phi(x, \lambda)$, $\theta(x, \lambda)$. *F. V. Atkinson.*

Volpato, Mario. Su un problema ai limiti relativo all'equazione $y''=f(x, y', y, \lambda)$. Giorn. Mat. Battaglini (4) 3(79), 132–143 (1950).

The existence of a solution $[\lambda_0, y_0(x)]$ of the eigenvalue problem

$$y''(x)=f[x, y(x), y'(x), \lambda], \quad y(a)=\alpha, \quad y(x_0)=\gamma, \quad y(b)=\beta,$$

where α, β, γ are arbitrary real constants, is proved under the following set of conditions. (1) The real-valued function $f(x, y, y', \lambda)$ is continuous with respect to (y, y', λ) and measurable with respect to x in the strip

$$S: \quad a\leqq x\leqq b, \quad |y|<\infty, \quad |y'|<\infty, \quad c<\lambda<d.$$

(2) In S, $p(x, \lambda)\leqq f(x, y, y', \lambda)\leqq q(x, \lambda)$, where $p(x, \lambda)$ and $q(x, \lambda)$ are summable with respect to x in the interval $a\leqq x\leqq b$ for each λ in the interval $c<\lambda<d$, and are continuous with respect to λ in $c<\lambda<d$. (3) $G(x_0, t)$ being defined as $(b-x_0)(t-a)/(b-a)$ for $a\leqq t\leqq x_0$ and as $(x_0-a)(b-t)/(b-a)$ for $x_0\leqq t\leqq b$, the integrals

$$\int_a^b G(x_0, t)p(t, \lambda)dt, \quad \int_a^b G(x_0, t)q(t, \lambda)dt$$

are non-decreasing functions of λ, tending to $-\infty$ and $+\infty$ as λ tends to c and d, respectively. The discussion includes some applications and variants of the main theorem.

L. A. MacColl (New York, N. Y.).

Guderley, Gottfried. A formula for the normalization constant in eigen value problems. Quart. Appl. Math. 10, 176–177 (1952).

Germay, R. H. Remarque sur l'extension d'un théorème de Poincaré aux solutions infiniment voisines des systèmes complètement intégrables d'équations aux différentielles totales renfermant un paramètre variable. Bull. Soc. Roy. Sci. Liège 20, 455–463 (1951).

For a passive Pfaffian system of two equations in n independent variables x_j, two dependent variables y, z and one parameter λ the coefficients of the differentials dx_j in the form solved for dy, dz are assumed power series in y, z, λ with coefficients of class C' in $x_1, \cdots, x_n$ and the solutions y, z are expressed as power series in λ whose coefficients are determined by certain integral equations.

J. M. Thomas (Paris).

Pini, B. Su certe questioni di periodicità e asintoticità per i sistemi lineari del primo ordine ai differenziali totali. Rend. Sem. Mat. Univ. Padova 20, 249–277 (1951).

This paper concerns the passive Pfaffian system

$$dy=A_v du+B_v dv,$$

where y is an unknown $n\times 1$ matrix and A, B are known $n\times n$ matrices. The A, B are bounded and measurable, A is absolutely continuous in v for every u, B is absolutely continuous in u for every v, and the derivatives A_v, B_u are integrable. For periodic A, B, the results of G. Floquet [Ann. Sci. École Norm. Sup. (2) 12, 47–88 (1883)] for ordinary equations are extended to the systems of this paper. For constant A, B the asymptotic behavior of the solutions under variation of the coefficients is discussed.

J. M. Thomas (Paris).

Scorza Dragoni, Giuseppe, e Volpato, Mario. Un teorema di unicità per le soluzioni di una equazione alle derivate parziali del primo ordine. Rend. Sem. Mat. Univ. Padova 20, 446–461 (1951).

A uniqueness theorem for $z_x=f(x, y, z, z_y)$ is given. The assumptions on f and on the solution z are numerous. A noteworthy feature is this: a solution, in the sense of this paper, may fail to satisfy the equation on a set whose orthogonal projection on the interval $0\leqq x\leqq l$ has measure zero. *J. M. Thomas* (Paris).

Pagni, Mauro. Un'osservazione sull'unicità della soluzione del problema di Cauchy per l'equazione

$$p=f(x, y, z, q).$$

Rend. Sem. Mat. Univ. Padova 20, 470–474 (1951).

An example is given to show that the system

$$z_x=f(x, y, z, z_y), \quad z(0, y)=0,$$

in which both f, z are subjected to a Lipschitz condition, can be satisfied almost everywhere by two distinct functions. The author notes that this contradicts the last clause in a paper by E. Bajada [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 11, 158–164 (1951); these Rev. 13, 656]. *J. M. Thomas* (Paris).

Cinquini Cibrario, Maria, e Cinquini, Silvio. Sopra una forma più ampia del problema di Cauchy per l'equazione $p=f(x, y, z, q)$. Ann. Mat. Pura Appl. (4) 32, 121–155 (1951).

This paper gives for the Cauchy problem $z_x=f(x, y, z, z_y)$, $z(0, y)=\phi(y)$ a treatment similar to that developed for the initial value problem for ordinary differential equations by Carathéodory [Vorlesungen über reelle Funktionen, Teubner, Leipzig-Berlin, 1918, Chap. XI, pp. 665–688]. Let I_b denote the interval $0\leqq x\leqq b$, and let R_n denote the space of n real variables. Let $g(x, y, z, q)$ belong to the class G_b $(b>0)$ if and only if: 1) g is defined on the product space $I_b\times R_3$; 2) with x fixed in I_b, g is continuous in (y, z, q) on R_3; 3) with (y, z, q) fixed in R_3, g is quasi-continuous in x on I_b; 4) for all (y, z, q) in R_3 and almost all x on I_b, $|g(x, y, z, q)|\leqq M(x)$ with $M(x)$ quasi-continuous and integrable (always Lebesgue) on I_b; 5) for all (y, z, q) and (y', z', q') in R_3 and almost all x on I_b,

$$|g(x, y', z', q')-g(x, y, z, q)| \leqq L(x)(|y'-y|+|z'-z|+|q'-q|)$$

with $L(x)$ quasi-continuous and integrable on I_b. [Presumably Tonelli's definition of quasi-continuity is used, which means, in this situation, that for each $\epsilon>0$ there exists a closed subset S_ϵ of I_b, with measure $>b-\epsilon$, such that the function is continuous on S_ϵ. Cf. Vitali e Sansone, Moderna teoria delle funzioni di variabile reale, 2nd ed., Zanichelli, Bologna, 1943, vol. I, p. 72 ff.; these Rev. 7, 376.] Let $\phi(y)$ belong to the class D if and only if $\phi(y)$ is defined and differentiable on R_1 and there exist constants k and l such that $|\phi'(y)|\leqq k$ and $|\phi'(y^*)-\phi'(y)|\leqq l|y^*-y|$ for y and y^* in R_1. Let $z(x, y)$ belong to the class K_b if and only if $z(x, y)$ is defined on $I_b\times R_1$ in such a way that z and z_y are continuous on $I_b\times R_1$ and z is absolutely continuous in x for each y in R_1. Let $z(x, y)$ belong to the class L_b if and only if z belongs to K_b, z_y is absolutely continuous in x for each y in R_1, and z and z_y are Lipschitzian in y uniformly in x.

Existence theorem: If $\phi(y)$ is in D and $f(x, y, z, q)$ has the property that f, f_y, f_z, and f_q are all in G_{a_0}, then there exists a number a with $0<a\leqq a_0$ and at least one function $z(x, y)$

in K_a which satisfies the equation

$$(*) \qquad z(x, y) = \phi(y) + \int_0^x f(\xi, y, z(\xi, y), z_y(\xi, y))d\xi;$$

moreover, $z(x, y)$ is in L_a and, for each y in R_1,

$$z_x(x, y) = f(x, y, z(x, y), z_y(x, y))$$

for almost all x in I_a. First uniqueness theorem: Under the hypotheses of the existence theorem, equation (*) cannot have two solutions in L_a. Second uniqueness theorem: Suppose that $f(x, y, z, q)$ has properties 1), 2), 3), and 4) required above in the definition of G_a, and that, for all (y, z, q) and (y, z', q') in R_3 and almost all x on I_a,

$$|f(x, y, z', q') - f(x, y, z, q)| \leqq H(x)|z' - z| + L(x)|q' - q|$$

with $H(x)$ and $L(x)$ quasi-continuous and integrable on I_a, and suppose also that $\phi(y)$ and $\phi'(y)$ are continuous on R_1; then equation (*) cannot have two solutions in the class K_a. The proofs apply Carathéodory's results to the equations of the characteristics, written as integral equations, and employ refined estimates, similar to those of Gronwall and Haar, for functions, difference quotients, and derivatives. The results justify the observation that even in the comparatively general situation discussed in this paper an integral surface is the locus of characteristic curves issuing from the initial curve. *F. A. Ficken* (Knoxville, Tenn.).

Germay, R. H. Sur la généralisation d'un théorème de Jacobi pour l'intégration des systèmes d'équations aux dérivées partielles du premier ordre de forme résolue par rapport aux dérivées en x_1. Bull. Soc. Roy. Sci. Liège **20**, 609–616 (1951).

The results of an earlier paper [Ann. Soc. Sci. Bruxelles. Ser. I. **65**, 103–108 (1951); these Rev. **13**, 559] are extended from 1 to k unknowns. *J. M. Thomas* (Paris).

Visconti, Antoine. Sur un type de calcul opératoriel applicable à la théorie des perturbations. C. R. Acad. Sci. Paris **234**, 1252–1255 (1952).

The solution of (1) $H(t)U = i(\partial/\partial t)U$ is considered. Here H is a linear bounded operator, $U(t, t_0)$ is a certain operator such that $U(t_0, t_0) = 1$. It is assumed that there exists a domain D_t in which (1) is equivalent to

$$(2) \qquad U(t, t_0) = 1 - i\int_{t_0}^{t} H(\tau)U(\tau, t_0)d\tau.$$

An integral operator is introduced which is associative, and distributive with respect to addition. It is shown how Picard's solution of successive approximations of (1) can be obtained formally in terms of this operator. It is indicated how the theory of this operator can be applied to several physical problems. *H. P. Thielman* (Ames, Iowa).

Hopf, Eberhard. The partial differential equation

$$u_t + uu_x = \mu u_{xx}.$$

Comm. Pure Appl. Math. **3**, 201–230 (1950).

[Title misprinted as $u_t + uu_x = \mu_{xx}$.] The author studies the quasi-linear differential equation given in the title, especially in the limit $\mu \to 0$, with the point of view of illustrating the behaviour of the solutions of the initial-value problems of hydrodynamics as viscosity and conductivity approach zero. This particular differential equation can be discussed explicitly because a transformation of variables reduces it to the linear problem of heat conduction. In particular, the behaviour of the solutions in the limit $t \to \infty$ with finite μ and in the limit $\mu \to 0$ are worked out. It is shown that the order of taking the two limiting processes is important. The limit function with $\mu \to 0$ is analogous to the solution for inviscid compressible fluid with shocks, and is, therefore, not completely determined by the differential equation with $\mu = 0$. A functional equation involving integrals is given, however, which allows the limiting process $\mu \to 0$ to be taken formally and still has all the limiting functions as solutions. *C. C. Lin* (Cambridge, Mass.).

Henrici, Peter. Bergmans Integraloperator erster Art und Riemannsche Funktion. Z. Angew. Math. Physik **3**, 228–232 (1952).

Elegant presentation of familiar material concerning the representation by means of Riemann's function of solutions of an elliptic partial differential equation in terms of analytic functions [Lewy, C. R. Acad. Sci. Paris **197**, 112–113 (1933); Bergman, Amer. J. Math. **70**, 856–891 (1948), p. 867; these Rev. **10**, 752]. *P. R. Garabedian.*

Kornhauser, E. T., and Stakgold, I. A variational theorem for $\nabla^2 u + \lambda u = 0$ and its applications. J. Math. Physics **31**, 45–54 (1952).

Die vorliegende Arbeit handelt über das, der partiellen Differenzialgleichung $\nabla^2 u + \lambda u = 0$ entsprechende erste und zweite Randwertproblem. Zugrundegelegt wird ein zweidimensionales einfach zusammenhängendes Gebiet R mit der Berandung B. Im Falle der ersten Randwertaufgabe $u|_B = 0$ seien die Eigenwerte bezeichnet mit $0 < \lambda_0 < \lambda_1 \leqq \cdots$. Im Fall der zweiten Randwertaufgabe $\partial u/\partial n|_B = 0$ seien die Eigenwerte bezeichnet mit $0 = \mu_0 < \mu_1 \leqq \cdots$. Ziel der Arbeit ist, zu zeigen, dass $\mu_1 < \lambda_0$. Für den Fall, dass der Bereich ein Kreis ist, sind ja die Eigenwerte bekannt. Bekannt ist ferner der von Lord Rayleigh auf Grund einer infinitesimalen Betrachtung aufgestellte und von mehreren Autoren exakt bewiesene Satz, dass unter allen am Rand eingespannten flächengleichen Bereichen dem Kreis der kleinste Wert von λ_0 entspricht. Hier stellen die Verf. auf Grund einer infinitesimalen Betrachtung den Satz auf, dass für die zweite Randwertaufgabe dem Kreis der grösste Wert von μ_1 entspricht. Die infinitesimale Betrachtung zerfällt in zwei Teile. Im ersten Teil werden kreisnahe Bereiche betrachtet. Im zweiten Teil wird gezeigt, dass sich jeder vom Kreis verschiedene Bereich bei gleichbleibendem Flächeninhalt so transformieren lässt, dass dabei μ_1 wächst. *P. Funk.*

Pólya, G. Remarks on the foregoing paper. J. Math. Physics **31**, 55–57 (1952).

Um den in der oben referierten Arbeit aufgestellten Satz zu beweisen, wird die Minimumeigenschaft von μ_1

$$\iint \phi dxdy = 0; \quad \mu_1 \leqq \iint (\phi_x^2 + \phi_y^2)dxdy \Big/ \iint \phi^2 dxdy$$

herangezogen. Verwendet man nun ein Koordinatensystem von der Beschaffenheit dass der Schwerpunkt des Bereiches mit dem Koordinatenursprung und die Achsen mit den Hauptachsen des Trägheitsellipsoids zusammenfallen und sei $I' = \iint x^2 dxdy \geqq \iint y^2 dxdy$ so ergibt sich, wenn man $\phi = x$ wählt $\mu_1 < A/I'$. Ferner ergibt eine einfache massengeometrische Betrachtung $I' \geqq A^2/4\pi$ wobei A den Flächeninhalt des Bereiches bedeutet. Daraus ergibt sich dann unter Verwendung des im vorstehenden Referat erwähnten Satzes von Rayleigh der erwünschte Satz. *P. Funk* (Wien).

Brownell, F. H. Spectrum of the static potential Schrödinger equation over E_n. Ann. of Math. (2) **54**, 554–594 (1951).

Let $V, u, w, \cdots$ denote real functions on $E_n: x=(x_1, \cdots, x_n)$, $r=|x|^{\frac{1}{2}}$, $L_p=L_p(E_n)$ and $\nabla=\partial^2/\partial x_1^2+\cdots+\partial^2/\partial x_n^2$. The first part of the paper is concerned with setting up relations between the formal differential operator $\mathbf{A}$, $\mathbf{A}w=\nabla w-V_w$, and certain self-adjoint operators on L_2. If $\alpha>0$, then there exists a positive, decreasing function $H_\alpha(r)$ of $r(>0)$ satisfying $\nabla H_\alpha=H_\alpha''+(n-1)r^{-1}H_\alpha'=\alpha^2H_\alpha$, which can be made unique by the requirement that $H_\alpha(r)$, has, up to a specified constant factor, the (n-dimensional) Fourier transform $(\alpha^2+r^2)^{-1}$. Let $K_\alpha(r)$ be defined as a constant multiple of $r^{-n+2}e^{-\alpha/4r}$, $(1+\log(1+\alpha^{-1}r^{-1}))(1+\alpha^{\frac{1}{2}}r^{\frac{1}{2}})e^{-\alpha r}$ or $\alpha^{-1}e^{-\alpha r}$ according as $n\geqq 3$, $n=2$ or $n=1$; so that $0<H_\alpha(r)<K_\alpha(r)$. Put $|V|_\alpha=$ ess. sup. $\int_{E_n}K_\alpha(|x-y|^{\frac{1}{2}})|V(y)|dy\cdots dy_n$. By treating the differential operator $\alpha^2\mathbf{I}-\mathbf{A}=(\alpha^2+V)-\nabla$ as a perturbation of $\alpha^2-\nabla$, which has $H_\alpha(|x-y|^{\frac{1}{2}})$ as a "Green kernel", the author shows that if $|V|_\alpha<1$, then there exists a bounded linear Hermitian integral operator $\mathbf{G}_\alpha$ on L_2 with the property that if u is in L_2 and L_1, then $w=\mathbf{G}_\alpha u$ is a "generalized" solution of $(\alpha^2\mathbf{I}-\mathbf{A})w=u$ in the following sense: if D denotes the set of functions φ having continuous second order partial derivatives and vanishing outside of a bounded set, then, in terms of scalar products, $(w, (\alpha^2\mathbf{I}-\mathbf{A})\varphi)=(u, \varphi)$ for all φ in D. Conversely, if u, w, Vw are of class L_2 and w is a generalized solution of $(\alpha^2\mathbf{I}-\mathbf{A})w=u$, then $w=\mathbf{G}_\alpha u$. Furthermore, $\mathbf{G}_\alpha$ satisfies Hilbert's identity for resolvents; there exists a set Σ of real λ-values, independent of α, such that the transformation $(\lambda+\alpha^2)^{-1}$ carries Σ onto the spectrum of $\mathbf{G}_\alpha$; and the spectral resolution of $\mathbf{G}_\alpha$ can be put into the form $\int(\lambda+\alpha^2)^{-1}dE(\lambda)$, where $E(\lambda)$ is independent of α. If $|V|_\alpha<1$ and D_0, the set of functions φ for which $V\varphi$ is in L_2, is dense in D, then $\alpha^2\mathbf{I}-\mathbf{G}_\alpha^{-1}$ is a self-adjoint extension (with spectrum Σ) of the operator $\mathbf{A}_0=\nabla-V$, whose domain is D_0 (this self-adjoint extension is not claimed to be unique).

The second half of the paper deals with relationships between the spectrum Σ and the behavior of the potential V (these are analogous to results known for the case $n=1$). One result is the following: if $\liminf_{B(c,d)}\int|V(x)|^2dx_1\cdots dx_n<\infty$, as $(c, d)\to(\infty, \infty)$, where $B(c, d)$ is the set

$$x_1>c,\ |x_2|\leqq d,\ \cdots,\ |x_n|\leqq d,$$

then Σ contains the half-line $0\leqq\lambda<\infty$. The author gives sufficient conditions to assure that no half-line $-\infty<\lambda<-\epsilon$ (<0) contains more than a finite number of points of Σ and conditions to assure that no point of $0<\lambda<\infty$ is in the point spectrum. *P. Hartman* (Baltimore, Md.).

Halilov, Z. I. On a method of solution of mixed problems. Doklady Akad. Nauk SSSR (N.S.) **83**, 659–662 (1952). (Russian)

The author considers a mixed problem for equations of the form

$$(1)\qquad L_{P;P}(u)+M_{P,t;P}(u)+N_{t;t}(u)=0,$$

where P is a point of a region D in n-space with boundary S and L, M, and N are linear differential operators. The operators depend on the variables preceding the semicolon and the differentiations are performed with respect to the variables following the semicolon. He seeks a solution subject to homogeneous boundary conditions (2) $R[u(t, Q)]=0$ for $Q\in S$ ($0\leqq t\leqq T$) and initial conditions $A[u(0, P)]=f(P)$ for $P\in D$. He gives a formal solution of (1) in the form $u(t, P)=\sum_{m=1}^{m}A_m(t)u_m(P)$, where the A_m's are the solutions of a certain infinite system of ordinary differential equations and the u_m's are the eigenfunctions of the differential operator L under the boundary conditions (2). He applies this method to the equation $u_t=u_{xx}-C(t, x)u$, where

$$u(0, t)=u(\pi, t)=0\ \ (0\leqq t\leqq T),\quad u(x, 0)=f(x)\ \ (0\leqq x\leqq\pi),$$

and $C(t, x)$ is continuous. *J. M. Danskin.*

Martin, M. H. A remark on characteristics. Proc. Amer. Math. Soc. **3**, 280–281 (1952).

One of the characteristics of the Tricomi equation $\psi_{\sigma\sigma}+\sigma\psi_{\theta\theta}=0$ is $9\theta^2=-4\sigma^3$, which is the envelope of the stream lines, $\psi=$const., of the Tricomi gas. The author is interested in similar relationships for the partial differential equation $L(u)=a_{ij}u_{x_ix_j}+b_iu_{x_i}=0$, $i, j=1, \cdots, n$, (repeated indices are summations from 1 to n) and finds them in the theorem: If $F(x_1, \cdots, x_n)=0$ defines implicitly a solution to $L(u)=0$, and if the one-parameter family of $(n-1)$-dimensional hypersurfaces $u=$const. in the space of the independent variables has an envelope upon which $F_{x_n}\neq 0$, the envelope is a characteristic hypersurface. According to a referee's note the theorem is contained in a theorem of Le Roux and Delassus as quoted by Hadamard in his book, Le problème de Cauchy et les équations aux dérivées partielles linéaires hyperboliques, Hermann, Paris, 1932, p. 102. *A. Gelbart* (Trondheim).

Malyužinec, G. D. The mathematical formulation of the problem of forced harmonic vibrations in an arbitrary region. Doklady Akad. Nauk SSSR (N.S.) **78**, 439–442 (1951). (Russian)

The author seeks a unified formulation of wave-propagation problems which shall ensure uniqueness for both finite and infinite regions, including the case in which the boundary extends to infinity. He lists very briefly shortcomings of the radiation principle, and of the principle of absorption [cf. A. G. Svešnikov, same Doklady **73**, 917–920 (1950); these Rev. **12**, 223]. More satisfactory he finds a variant of the latter principle used by V. Fock [Ann. Physik (5) **17**, 401–420 (1933)], based on analytic continuation from imaginary values of the propagation constant. He goes on to prove a uniqueness theorem for an appropriate Green's function for $\Delta v+\lambda v=0$, where $\pi/2<\arg\lambda<3\pi/2$, which can be deduced simply from a theorem of the author's [same Doklady **78**, 229–230 (1951); these Rev. **13**, 350], and concludes with miscellaneous remarks. *F. V. Atkinson.*

Bureau, Florent. Intégrales de Fourier et problème de Cauchy. Ann. Mat. Pura Appl. (4) **32**, 205–233 (1951).

The reviewer [Quart. J. Math., Oxford Ser. (2) **1**, 123–135 (1950); these Rev. **12**, 184] has given a method for using Fourier transforms rigorously in the solution of the wave equation, finding this solution explicitly for an odd number, and for two, space dimensions. The present article gives a method applicable to any number of dimensions, based on the use of finite parts and logarithmic parts of integrals, and on an extension of Parseval's formula to finite or logarithmic parts. The ordinary and the damped wave propagation equations are discussed. *J. L. B. Cooper.*

Vallander, S. V. On integration of an hyperbolic system of two equations with two independent variables. Doklady Akad. Nauk SSSR (N.S.) **83**, 637–639 (1952). (Russian)

Consider the hyperbolic system of two equations

$$\frac{\partial y}{\partial\eta}=A(\xi, \eta)\frac{\partial x}{\partial\eta},\quad \frac{\partial y}{\partial\xi}=B(\xi, \eta)\frac{\partial x}{\partial\xi}.$$

It is shown that if A and B can be put into a specified form then the general solution for x is representable in the form $x=f_1(\xi,\eta)\Phi_1(\xi)+f_2(\xi,\eta)\Phi_2(\eta)$, where f_1 and f_2 are determinable functions and Φ_1 and Φ_2 are arbitrary functions of their arguments. *C. G. Maple* (Washington, D. C.).

Zwirner, Giuseppe. Teoremi di unicità e di confronto per gli integrali di una particolare classe di equazioni differenziali a derivate parziali del secondo ordine. Rend. Sem. Mat. Univ. Padova **20**, 329–345 (1951).

For the system

(1) $z_{xy}=f(x,y,z)$, $z(0,y)=\psi(y)$, $z(x,0)=\varphi(x)$, $\varphi(0)=\psi(0)$

uniqueness and comparison theorems are given. The first uniqueness theorem assumes (i) f continuous, (ii) f bounded, (iii) f non-increasing in z, (iv) φ, ψ of class C', all for (x,y) in a closed rectangle and z finite. The second replaces hypothesis (iii) by the existence of a function $F(x,z_2-z_1)$ conditioned, among other things, by (v) for some K, $0\leqq z_2-z_1\leqq K$ implies $f(x,y,z_2)-f(x,y,z_1)\leqq F$; (vi) the behavior of the maximum solution of an integral equation involving F. The third replaces (iii) by (vii) $z_1\leqq z_2$ implies

$$[f(x,y,z_2)-f(x,y,z_1)]x\delta_1\leqq z_2-z_1,$$

the rectangle being $0\leqq x\leqq\delta$, $0\leqq y\leqq\delta_1$. The first comparison theorem is: if f, f_1 satisfy (i), (ii) and (viii)

$$f(x,y,z_2)-f(x,y,z_1)\leqq f_1(x,y,z_2-z_1),$$

then in the rectangle $0\leqq G(x,y)-g(x,y)\leqq\Phi(x,y)$, where G, g are the maximum, minimum solutions of (1) and Φ is the maximum solution of $z_{xy}=f_1$, $z(0,y)=z(x,0)=0$. The second comparison theorem, in addition to the satisfaction of (i), (ii), (iii), assumes that f satisfies an inequality involving a double integral and gives an inequality $\omega(x,y)\leqq G(x,y)$ on the maximum solution G. *J. M. Thomas* (Paris).

Mertens, Robert. Sur la diffusion multiple de particules chargées. C. R. Acad. Sci. Paris **233**, 856–858 (1951).

The author considers by a standard method the equation of radiative transfer in semi-infinite plane-parallel atmospheres with a law of scattering which can be expressed as a series in Legendre polynomials. [The method of solution described is well-known and the various formulae given here are equivalent to those which can be found for example in S. Chandrasekhar, Radiative Transfer, Oxford, 1950, chap. VI, §48 [these Rev. **13**, 136].] *S. Chandrasekhar.*

Hornich, Hans. Zur Theorie der partiellen Differentialgleichungen. Math. Ann. **124**, 148–150 (1952).

The author shows the regularity of solutions of equations such as $\alpha^r\,\partial^r f/\partial z^r-\partial^r f/\partial w^r=g(x,y)$, $z=\log x$, $w=\log y$ in a neighborhood of $x=y=0$ depends on the arithmetical character of the parameter α, assuming $g(x,y)$ is analytic in an including neighborhood [cf. the review of the author's paper in Monatsh. Math. **54**, 183–187 (1950); these Rev. **12**, 708; besides the author's reference to Bourgin and Duffin cited in that review, a greater variety of cases arises in Bourgin, Duke Math. J. **7**, 97–120 (1940); these Rev. **2**, 290]. *D. G. Bourgin* (Urbana, Ill.).

Sibirani, Filippo. Su alcune classi di equazioni alle derivate parziali delle quali può costruirsi l'integrale completo. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) **8** (1950–51), 91–94 (1952).

This paper contains four theorems, each of which gives a complete integral of a partial differential equation of a special type when the general integral of an associated ordinary differential equation is known. For example, the third theorem states that if $y=\chi(x,c_1,c_2,\cdots,c_{2n})$ is the general integral of $f(y,y_2,y_4,y_6,\cdots,y_{2n})=0$ where $y_s=d^sy/dx^s$, and $c_1,c_2,\cdots,c_{2n}$ are arbitrary constants, then $f(z,\partial^2z/\partial x\partial y,\partial^4z/\partial x^2\partial y^2,\cdots,\partial^{2n}z/\partial x^n\partial y^n)=0$ has a complete integral $z=\chi(c_{2n+1}x+y/c_{2n+1},c_1,c_2,\cdots,c_{2n})=0$ where c_{2n+1} is an arbitrary constant. *E. T. Copson.*

Functional Analysis, Ergodic Theory

Leĭbenzon, Z. L. On the separation of convex sets by a hyperplane in linear topological spaces. Uspehi Matem. Nauk (N.S.) **7**, no. 2(48), 165–167 (1952). (Russian)

The author proves that if E is a linear topological space, G_1 and G_2 are disjoint convex subsets of E, and G_1 has an interior point, then G_1 and G_2 can be separated by a hyperplane. His proof is the same as those of M. H. Stone [Convexity, Chicago, 1946 (mimeographed lecture notes)] and the reviewer [Duke Math. J. **18**, 443–466 (1951); these Rev. **13**, 354], using the fact, due to Kakutani [Proc. Imp. Acad. Tokyo **13**, 93–94 (1937)], that E can be covered by disjoint convex sets $E_i\supset G_i$ and then showing that $\bar{E}_1\cap\bar{E}_2$ is the desired hyperplane. *V. L. Klee, Jr.* (Princeton, N. J.).

Mil'man, D. The facial structure of a convex bicompact space and integral decompositions of means. Doklady Akad. Nauk SSSR (N.S.) **83**, 357–360 (1952). (Russian)

This note applies results on coverings with special properties to the study of extreme points; notation and definitions follow those of the author's earlier notes [same Doklady **57**, 119–122 (1947); **59**, 1045–1048 (1948); **60**, 25–27 (1948) (with D. A. Rutman); these Rev. **9**, 192, 449, 448]. K is a bicompact convex space with functionally determined topology; A is its set of extreme points, and Γ is the closure of A in K. A set X in K is called a facial set if along with every interior point of a segment of K it contains the whole segment. It is shown that there is a topology of A, the γ-topology, in which A is a bicompact T_1-space and the closed sets are the intersections of A with the facial sets of K. Relations are given between normalized Borel measures on Γ and somewhat more general Borel measures on A (in its γ-topology).

For each f_0 in K there is a Γ-minimum closed set $\Gamma(f_0)$ in Γ (a γ-minimum γ-closed set $\gamma(f_0)$ in A) such that f_0 is in the closed convex hull of $\Gamma(f_0)$ (of $\gamma(f_0)$). To each f_0 in K corresponds a normalized Borel measure σ_0 on $\Gamma(f_0)$. By a topological and measure-theoretical decomposition theorem there exist reduced canonical coverings $\{F_i, i\leqq n\}$ of $\Gamma(f_0)$ with special properties, among them that $\sigma_0(F_i\cap F_j)=0$ if $i\neq j$. It is shown also that to each canonical closed subset F of Γ there is an $f_F=[\sigma_0(F)]^{-1}\int_F f d\sigma_0(I_f)$ such that F is $\Gamma(f_F)$. This scattering of results yields an integral decomposition of f_0 over $\Gamma(f_0)$ in which the elementary sets of the decomposition are themselves Γ-minimum sets of points of K. Parallel results for A in its γ-topology are given. *M. M. Day.*

Klee, V. L., Jr. Invariant metrics in groups (solution of a problem of Banach). Proc. Amer. Math. Soc. **3**, 484–487 (1952).

G. Birkhoff-Kakutani's (one-sided or two-sided) invariant metrization theorem for a Hausdorff group is sharpened so as to imply an affirmative answer to a question of S. Banach

[Théorie des opérations linéaires, Warsaw, 1932, p. 132]: Every complete linear metric space can be metrized as a (complete) space of type (F). Also it is proved that a normed linear space is a Banach space if and only if it is topologically complete. *K. Yosida* (Nagoya).

Klee, V. L., Jr. Convex sets in linear spaces. II. Duke Math. J. **18**, 875–883 (1951).

The paper continues a previous article [same J. **18**, 443–466 (1951); these Rev. **13**, 354]. Some results are given for polyhedral connectedness of the complement of a convex cone in a linear system L. The properties given are formally the same as those for finite-dimensional Euclidean spaces. The next section enumerates characterizations of hyperplanes according to (order) separation properties and topological properties and is succeeded by a section on the covering of Banach spaces by hyperplanes including remarks on the continuum hypothesis. The Dieudonné example [Revue Sci. **81**, 277–278 (1943); these Rev. **7**, 124] of l_1 as a space not satisfying Tukey's convex set separation theorems [Portugaliae Math. **3**, 95–102 (1942); these Rev. **4**, 13] is generalized by showing l_1 may be replaced by any separable nonreflexive Banach space. A joint argument with Erdös shows the real numbers R may be assigned a Hausdorff topology under which points are not open, though R is disconnected, and such that translations and multiplications by a nonzero real preserve openness.

D. G. Bourgin (Urbana, Ill.).

Dieudonné, Jean. Complex structures on real Banach spaces. Proc. Amer. Math. Soc. **3**, 162–164 (1952).

Un espace de Banach complexe peut être considéré comme un espace réel d'une façon évidente. Soit maintenant E un espace réel. S'il existe un automorphisme $u(x)$ de E tel que $u^2(x)=-x$ il est possible d'introduire en E une structure d'espace complexe en définissant $(\lambda+i\mu)x=\lambda x+\mu u(x)$. L'auteur démontre qu'en général les espaces réels à une infinité de dimensions (le cas de dimension finie étant facilement résolu) ne possèdent aucun automorphisme $u(x)$ tel que $u^2(x)=-x$. En particulier, l'espace introduit par R. C. James [Ann. of Math. **52**, 518–527 (1950); ces Rev. **12**, 616] est dépourvu de cette propriété. La démonstration se base sur deux faits: (1) Soient E_0 un espace à structure complexe, E le même espace considéré comme espace réel. Soient E', E_0' leurs espaces duaux. Il existe un isomorphisme $v(x)\to w(x)$ de E' sur E_0' défini par $w(x)=v(x)-iv(ix)$. À son tour, cet isomorphisme donne lieu à un isomorphisme Φ de E'' sur E_0'' (le dual de l'espace complexe E_0'). (2) Pour l'espace E de James, E''/E est de dimension réelle égale à 1. S'il est possible de considérer E comme ayant une structure complexe E_0, alors d'après (1) et (2), E_0''/E_0 est de dimension réelle égale à 1, tandis que E_0''/E_0, étant un espace complexe, doit être de dimension réelle au moins 2.

E. R. Lorch (New York, N. Y.).

Nakamura, Masahiro. A proof of the Hahn-Birkhoff theorem. Notes on Banach space. X. Proc. Japan Acad. **26**, no. 10, 9–10 (1950).

The earlier proof [Nakamura, Tôhoku Math. J. (2) **1**, 100–108 (1949); these Rev. **11**, 186] of G. Birkhoff's theorem on the Hahn-decomposition of linear functionals is here simplified by the use of Zorn's Lemma.

R. Arens (Los Angeles, Calif.).

Sz.-Nagy, Béla. Perturbations des transformations linéaires fermées. Acta Sci. Math. Szeged **14**, 125–137 (1951).

Let ${}_\lambda T=T-\lambda I$ where T is a distributive closed operation on a dense domain D of the Banach space B to B. If λ is in the spectrum of T, then its principal multiplicity is the dimension of the subspace spanned by the solutions of $\lim_{n\to\infty}\|{}_\lambda T^n x\|^{1/n}=0$. Suppose $T=T_0$ and $T(\epsilon)$ is the convergent series $\sum\epsilon_i T_i$. A fundamental tool is the observation that $T(\epsilon)$ is closed if $\|T_\lambda x\|$ is inferior to the kth root of a fixed linear combination of the norms of $\|x\|$ and $\|Tx\|$. The case of λ isolated and of principal multiplicity 1 yields a theorem on the perturbations $\lambda(\epsilon)$, $T(\epsilon)$, stating analyticity in ϵ and the principal multiplicity 1 for $\lambda(\epsilon)$, entirely analogous to results for the self-adjoint operators in Hilbert space. For a principal multiplicity larger than 1 the theorem need not be true and a simple example is given of a perturbation $\lambda(\epsilon)$ which is no longer analytic in ϵ in any neighborhood of λ_0. *D. G. Bourgin* (Urbana, Ill.).

Kreĭn, M. G., and Krasnosel'skiĭ, M. A. Stability of the index of an unbounded operator. Mat. Sbornik N.S. **30**(72), 219–224 (1952). (Russian)

Let A be a linear operator from one Banach space E_1 into another E_2 which is closed, the set of whose values $\mathfrak{R}(A)$ is a closed subspace, the set of whose zeros $\mathfrak{N}(A)$ has finite dimensionality $\alpha(A)$, and such that the factor-space $E_2/\mathfrak{R}(A)$ has finite dimensionality $\beta(A)$. The authors prove for the index $\nu(A)=\alpha(A)-\beta(A)$ two results in extension of results of the reviewer [Mat. Sbornik N.S. **28**(70), 3–14 (1951); these Rev. **13**, 46, where the notation $\kappa(A)$ was used for $\nu(A)$]. The first is that there is a positive ρ such that for bounded B with $\|B\|<\rho$, $A+B$ satisfies the same conditions as A, with $\nu(A+B)=\nu(A)$, $\alpha(A+B)\leqq\alpha(A)$; the novelties here are that A need not be bounded, E_1 need not be the same as E_2, and the result for $\alpha(A)$. The second result states that $\nu(A+B)=\nu(A)$ if B is bounded and finite-dimensional, or again if B is completely continuous and E_2 has a basis; in this case the extension is partial in that the reviewer did not require the existence of a basis. The authors speculate that the basis requirement is not essential. Full proofs are given, on different lines to those of the reviewer. There is close connection with the authors' joint work with D. P. Mil'man [Sbornik Trudov Inst. Mat. Akad. Nauk Ukrain. SSR **11**, 97–112 (1948), not accessible to the reviewer].

F. V. Atkinson (Ibadan).

Hattori, Isami. On ergodic theorem in reflexive spaces. Mem. Fac. Sci. Kyūsyū Univ. A. **6**, 17–19 (1951).

The author proves that if T is a bounded linear transformation in a reflexive space B with $|T|\leqq 1$, then, for all $x\in B$, $(1/N)\sum_0^{N-1}T^n x$ converges strongly. The reviewer regrets to have to record that neither the result nor the method of proof may be regarded as new [Lorch, Bull. Amer. Math. Soc. **45**, 945–947 (1939); these Rev. **1**, 242].

E. R. Lorch (New York, N. Y.).

Owchar, Margaret. Wiener integrals of multiple variations. Proc. Amer. Math. Soc. **3**, 459–470 (1952).

Let C be the space of real-valued functions $x(t)$ continuous on $0\leqq t\leqq 1$ and vanishing at $t=0$. The first variation $\delta F(x|y)$ of a functional $F(x)$ defined over C is defined as $d[F(x+hy)]/dh|_{h=0}$ and the nth variation is defined as the first variation of the $(n-1)$st variation. R. H. Cameron [Proc. Amer. Math. Soc. **2**, 914–924 (1951); these Rev. **13**, 659] has given a formula for the integral over C of the first

variation $\delta F(x|y_0)$ for y_0 a fixed absolutely continuous function with a derivative $y_0'(t)$ which is essentially a bounded variation, and F a suitably restricted functional. Using this result, the author obtains by induction a formula for the integration of the nth variation. Similarly she obtains a theorem on the integral of a functional related to the nth variation. As a special case of her first theorem she evaluates the Wiener integral over C of the product $\prod_{i=1}^{2n}\int_0^1 x_i(t)dx(t)$ where the x's are functions of bounded variation on (0, 1). For $n=1$ this formula was given by Wiener [Acta. Math. 55, 117–258 (1930)]. *W. T. Martin.*

Ōhira, Keishirō. On a certain complete, separable and metric space. Mem. Fac. Sci. Kyūsyū Univ. A. 6, 9–15 (1951).

The author considers a complete separable metric space R satisfying the following generalization of the von Neumann-Jordan condition characterizing Hilbert spaces among normed linear spaces: For each pair of points B, C, of R there exists a point P of R satisfying the conditions $\overline{AB}^2+\overline{AC}^2=2\overline{AP}^2+\overline{BP}^2+\overline{CP}^2$ for each A in R. It is shown that R is isometric to a closed convex subset of Hilbert space. *M. M. Day* (Urbana, Ill.).

***Aronszajn, N. Applied functional analysis.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 123–127. Amer. Math. Soc., Providence, R. I., 1952.

This report consists of a few general Hilbert space interpretations for partial differential systems (A, Λ_i) defined over a Euclidean domain D with boundary C as (1) $Av=0$, $\Lambda_i v|_C=\psi_i$, $i=1, \cdots, N$. Under proper regularity conditions and restrictions on A and Λ_i, (A, Λ_i) can be considered a self-adjoint operator on a suitable Hilbert space. Two usual representations of this sort are mentioned. In the first, the Hilbert space consists of the subset of $L_2(D)\cdot C^m(D)$ satisfying $\Lambda_i v=0$, where the equivalence of (1) with a non-homogeneous equation with vanishing boundary conditions is used. The second scheme is restricted to elliptic or parabolic equations and requires the existence of an identity

$$(2)\qquad \int_D Av\bar{v}d\tau=\sum\int_D |B_i v|^2 d\tau+\sum\int_C |\Omega_i v|^2 ds.$$

The right hand side of (2) defines a natural quasi norm. Under adjunction of terms in $\int|\Lambda_i v|^2 ds$ or other restrictions one may get a true norm. The subset of elements in $C^{2n}(D)$, where n is the order of A, constitutes an incomplete Hilbert space in this norm. *D. G. Bourgin* (Urbana, Ill.).

Putnam, C. R. The spectra of quantum-mechanical operators. Amer. J. Math. 74, 377–388 (1952).

The author investigates Hermitian operators P and Q on a Hilbert space H satisfying the commutation relation $(PQ-QP)x=ix$ for all vectors x in a domain D under such weak regularity conditions that they are not necessarily unitarily equivalent to the well-known Schrödinger operators. Among other results, if P is Hermitian ($P\subset P^*$ and has dense domain), then P cannot have a real eigenvalue with eigenfunction in D. *I. E. Segal* (Chicago, Ill.).

Kurth, Rudolf. Zum Ergodenproblem. Z. Angew. Math. Physik 3, 232–235 (1952).

The author deduces a trivial consequence of the Birkhoff ergodic theorem (a consequence that is frequently even stated as a part of the theorem). By an immediate application of the Schwarz inequality he then proves that the dispersion of the time mean of a function is not larger than that of the function itself, and shows by means of an elementary example that both this estimate and its specialization to characteristic functions are best possible. *P. R. Halmos* (Chicago, Ill.).

Oxtoby, John C. Ergodic sets. Bull. Amer. Math. Soc. 58, 116–136 (1952).

Let T be a homeomorphism of a compactum Ω. A measure μ on Ω, invariant under T and $\mu(\Omega)=1$, is called ergodic if the dynamical system (Ω, T, μ) is ergodic. Kryloff and Bogoliouboff [Ann. of Math. (2) 38, 65–113 (1937)] gave a method of construction of all the possible ergodic measures for (Ω, T). Ergodic sets stand in one-to-one correspondence with the ergodic measures, and each ergodic measure vanishes outside the corresponding set. The author gives a comprehensive review, with a number of remarks and simplifications of the works (due to Ambrose, Fomin, Halmos, Hedlund, Kakutani, Morse, von Neumann, Oxtoby-Ulam, Robbins, Yosida, etc.) which center around the theory of ergodic sets. A self-sufficient development of the basic theorems of Kryloff and Bogoliouboff are given here in only 4 pages. A simple characterisation of transitive points and the conditions under which the ergodic theorem holds uniformly are obtained. Also an extension to a non-compact system is considered. Finally, by considering the shift transformation on a sequence space, there is given an example of a minimal set that is not strictly ergodic. *K. Yosida.*

Êl'sgol'c, L. E. An estimate for the number of singular points of a dynamical system defined on a manifold. Amer. Math. Soc. Translation no. 68, 14 pp. (1952).

Translated from Mat. Sbornik (N.S.) 26(68), 215–223 (1950); these Rev. 11, 671.

Theory of Probability

***van Dantzig, D. Sur l'analyse logique des relations entre le calcul des probabilités et ses applications.** Congrès International de Philosophie des Sciences, Paris, 1949. Vol. IV, Calcul des probabilités, pp. 49–66. Actualités Sci. Ind., no. 1146. Hermann & Cie., Paris, 1951. 860 francs.

The author takes a purely objective attitude towards probability, and divides the problem of its applications into four parts. (1) the mathematical system; he uses the familiar measure-like postulates. (2) The model; he takes a finite collective, with symmetry assumptions replacing those of disorder. (3) Getting from the empirical data to the model ("embrayage"). (4) Getting from the results thereby calculated to statements which can be made concerning experimental date ("débrayage"). The author first attacks (4), and suggests a modification of Cournot's Principle (i.e., that "events of very small probability will not occur"). He realizes the difficulty produced by combining this principle with the additivity of the probability of many disjoint unprobable events, so he serves the principle up in a new form, depending on making various combinations of a large set of predictions, so that the principle need be applied only once for the whole set and additivity need not be used. The reviewer can see neither clarity nor merit in this scheme. Problem (3) is based essentially on a "maximum likelihood"

method, keeping only models and parameter values leading to maximum probability of obtaining the data actually observed. The ambiguity seems to trouble the author. The author admits that his scheme is not particularly satisfactory, but sees no better way at present of giving an objective account of the subject, and hopes the discussion will be of some help. *B. O. Koopman* (New York, N. Y.).

***de Finetti, B. Rôle et domaine d'application du théorème de Bayes selon les différents points de vue sur les probabilités.** Congrès International de Philosophie des Sciences, Paris, 1949. Vol. IV, Calcul des probabilités, pp. 67–82. Actualités Sci. Ind., no. 1146. Hermann & Cie., Paris, 1951. 860 francs.

The author argues for the subjective conception of probability and the use of Bayes' theorem in statistical estimation, holding that the latter position is close to the former. His chief points are that the objective definitions require the subjective for their application, that their proponents use, but leave vague and unformulated, the subjective notion, that its formulation is possible and when done renders the objective definition unnecessary, reducing its procedures to mere methods of measuring probability in important special cases. He holds that, contrary to what is often said, the subjective notion is non-metaphysical, being merely the attempt to describe and systematize the thinking of common sense. These ideas are presented in connection with the use of Bayes' theorem, which is claimed to be, in principle, at the basis of all theories of statistical estimation, whether they acknowledge it or not. All the arguments have been advanced many times before; but they are so frequently ignored that it is thought worth while to repeat them.

B. O. Koopman (New York, N. Y.).

***de Finetti, Bruno. Recent suggestions for the reconciliation of theories of probability.** Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 217–225. University of California Press, Berkeley and Los Angeles, 1951. $11.00.

This paper discusses two of the great controversial points in the theory of probability and statistical estimation: the objective versus the subjective conception of probability, and the modern decision theory versus the Bayes' method of estimation. His discussion of the first point is given briefly at the end of the paper, where he indicates his reasons for favoring a properly interpreted subjective conception and differs from some recently attempted compromises aimed at admitting a dualism. His chief point is that an objective theory requires something equivalent to a subjective admission at the moment of its concrete application. He mentions the possibility of non-numerical subjective formulations in terms of order, but appears unacquainted with the reviewer's precise mathematical formulation in terms of partially ordered systems. The discussion of estimation emphasizes the two traps into which both schools have so often fallen: The modern statistical estimators, of believing that their theory can ever be applied to practical problems without ad hoc admission of a notion equivalent to a priori probability; and the followers of Bayes' method, of tending to regard total absence of knowledge as necessarily implying equal a priori probabilities. He gives his correct version of the way in which methods of estimation should be understood: A procedure shown to be advantageous for all possible a priori distributions should be the one chosen.

B. O. Koopman (New York, N. Y.).

***Fortet, R. Faut-il élargir les axiomes du calcul des probabilités?** Congrès International de Philosophie des Sciences, Paris, 1949. Vol. IV, Calcul des probabilités, pp. 35–47. Actualités Sci. Ind., no. 1146. Hermann & Cie., Paris, 1951. 860 francs.

The author discusses certain questions concerning probability and its application; he throws light on these without attempting to reach dogmatic conclusions; in fact, the question in the title of the paper is never answered. First, after giving the usual measure-like axioms, he shows that examples in quantum mechanics which have been cited as violating the laws of probability do, in fact, nothing of the kind: if the notion of event as implying the actual performance of an experiment is used, there is no violation. This leads him to stress, with examples, the need for a full definition of "event." The question of whether events necessarily have probabilities is touched on; some examples indicating that they do not are interpreted as merely indicating that they are not definable as events. Statistical estimation and the use of Bayes' rule are considered, the latter being favored; and the possibility of a new axiom concerning the existence of a priori probabilities is mentioned. Finally, in connection with the difficulties of the a priori notion, the author considers the possibility of a more flexible non-numerical probability that might be a better rendering of intuition and that might lead to upper and lower numerical probabilities (as outer and inner measure). He says that it would be desirable to have this matter investigated—apparently quite ignorant of the fact that such an investigation was carried out by the reviewer over twelve years ago [Ann. of Math. **41**, 269–292 (1940); these Rev. **1**, 245].

B. O. Koopman (New York, N. Y.).

Collinder, Björn. La règle de succession dans le calcul des probabilités. Från filosofiens och forskningens fält. Nya rön inom skilda vetenskaper, pp. 138–156, Upsala, 1950.

A quasi-philosophical discussion of the application of Bayes' theorem in prediction, often seeming to make use of something equivalent to the principle of sufficient reason. Criticisms and counter-criticisms are discussed. Contains nothing that has not been said before. *B. O. Koopman.*

Searle, S. R. Probability—the difficulties of definition. J. Inst. Actuaries Students' Soc. **10**, 204–212 (1951).

The three definitions of probability: counting chances (Laplace), frequency (von Mises), and subjective (Keynes), are considered, and the usual difficulties indicated. The author contributes nothing new and seems unfamiliar with recent literature. He seems to be unaware of the difference between the problem of formulating the rules of consistency between probability statements (the laws of probability), and the problem of finding rules which lead unequivocally to the initial probability statements. *B. O. Koopman.*

Bizley, M. T. L. Some notes on probability. J. Inst. Actuaries Students' Soc. **10**, 161–203 (1951).

This is a critical survey of the foundations of probability. The author starts with the question of how this term is to be defined, which raises at once the following issues (quoted): "First, is the definition to be subjective? By a subjective definition we shall mean in this paper one that admits the possibility that two persons in the same state of relevant knowledge may reasonably hold different views as to the probability of a given event. Secondly, is probability to be only a theoretical concept, or must our definition be such that a probability can be determined by practical experi-

ments and applied to practical problems? Thirdly, are we to embrace all types of event or only a restricted class, e.g. those that occur or can be imagined to occur on a large number of occasions? Fourthly, shall we define probability as a number or are we to admit the existence of a non-numerical probability? And if we stipulate that it is a number, must it be a rational one?"

The author then examines various existing theories in the light of these questions, and finds that while each has something to contribute, none of them is satisfactory; nor does he seem to see any prospect of deriving a satisfactory theory in the present state of our knowledge. No new arguments appear.

While clear and useful as an approach to the subject at an early level, the treatment does not appear to be aware of the deeper issues. Thus the author's definition of a "subjective" theory turns on the notion of "relevant" information as a self-explanatory concept. He seems not to realize that his words if taken literally would make every theory subjective, as soon as it faces any application. Nor does he perceive the nature and magnitude of the assumption that a treatment complying with his apparent desiderata is possible in principle. *B. O. Koopman.*

Kalinowski, Walbert C., and Regan, Francis. A postulational treatment of the probability for certain types of emissions. Math. Mag. **25**, 175–181 (1952).

The authors generalize results given by Fry [Probability and its engineering uses, Van Nostrand, New York, 1928] for the probability of emissions of radioactive material in various time intervals. Their results are all instances of the following multi-variable generating function for emissions in successive intervals $t_1, t_2, \cdots, t_m$ with N original particles and $kn(t)dt$ the probability of an emission in dt when $n(t)$ particles are available

$$P(x_1, x_2, \cdots, x_m) = \left[1+\sum_{j=1}^{m} q(t_1)\cdots q(t_{j-1})p(t_j)(x_j-1)\right]^N$$

where $q(t)=1-p(t)=e^{-kt}$. E.g., for emissions in intervals t_2, t_4 of the succession t_1, t_2, t_3, t_4, the generating function $P(1, x, 1, x)$ is binomial with $p=q(t_1)[p(t_2)+q(t_2)q(t_3)p(t_4)]$.

J. Riordan (New York, N. Y.).

***Dvoretzky, A., and Erdös, P. Some problems on random walk in space.** Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 353–367. University of California Press, Berkeley and Los Angeles, 1951. $11.00.

Let $P_1, \cdots, P_d$ be an orthogonal system of unit vectors in a d-dimensional Euclidean space R_d, and let

$$\{X_d(n, \omega) \mid n=1, 2, \cdots\}$$

be an infinite sequence of mutually independent R_d-valued random variables with the same distribution:

$$\Pr\{X_d(n, \omega) = \pm P_i\} = (2d)^{-1}, \quad i=1, \cdots, d.$$

Then $S_d(n, \omega)=\sum_{k=1}^{n} X_d(k, \omega)$ may be considered as a random walk in R_d. Let $L_d(n, \omega)$ be the number of different vectors in the first n terms $S_d(1, \omega), \cdots, S_d(n, \omega)$, and let $E_d(n)$ be the expected value of $L_d(n, \omega)$. The authors first prove that

$$E_2(n) = \frac{\pi n}{\log n} + O\left(\frac{n \log\log n}{(\log n)^2}\right),$$

$$E_3(n) = n\gamma_3 + O(\sqrt{n}),$$

$$E_4(n) = n\gamma_4 + O(\log n),$$

$$E_d(n) = n\gamma_d + \beta_d + O(n^{3-d/2}) \quad (d \geqq 5),$$

where $\gamma_d (d \geqq 3)$, $\beta_d (d \geqq 5)$ are positive constants. Then the following strong law of large numbers is obtained for $d \geqq 2$:

$$\Pr\left\{\lim_{n\to\infty} \frac{L_d(n, \omega)}{E_d(n)} = 1\right\} = 1.$$

The proof of this result for the case $d=2$ is much more difficult than for the case $d \geqq 3$. (There is no analogous result for the case $d=1$.) These results are closely related with the results on the Hausdorff measure of the paths of a Brownian motion discussed by P. Lévy [Amer. J. Math. **62**, 487–550 (1940); C. R. Acad. Sci. Paris **233**, 600–602 (1951); these Rev. **2**, 107; **13**, 363].

Let $\|S_d\|$ be the length of a vector S_d in R_d. From a well-known result of G. Pólya [Math. Ann. **84**, 149–160 (1921)] follows that $\Pr\{\|S_d(n, \omega)\| \to \infty\} = 1$ for $d \geqq 3$ while $\Pr\{S_d(n, \omega) = 0$ for infinitely many $n\} = 1$ for $d=2$. The authors show that if $g(n)$ is a positive-valued function with $g(n) \downarrow 0$ then $\Pr\{\|S_d(n, \omega)\| < g(n)\sqrt{n}$ for infinitely many $n\} = 0$ or 1 according as the series $\sum_{n=1}^{\infty} (g(2^n))^{d-2}$ is convergent or divergent ($d \geqq 3$). The proof is based on the idea of approximating discrete random walk by Brownian motion.

S. Kakutani (New Haven, Conn.).

Lauwerier, H. A. A linear random walk with a partly reflecting partly absorbing barrier. Appl. Sci. Research B. **2**, 294–300 (1952).

The simplest symmetric random walk in a one-dimensional lattice is studied under the assumption that at a given fixed point the particle is absorbed or reflected with probabilities p or $1-p$, respectively. The probability $h_n(z)$ of being at the point z after n steps is expressed as a contour integral by means of the generating function and Cauchy's formula. An asymptotic formula for large n is obtained by using the method of steepest descent. The passage to the limit of a continuous diffusion process is discussed. [The nature of this passage to the limit is not stated by the author with sufficient precision. For instance, the step length l occurring in the author's final expression is actually zero in the limit, as $n \to \infty$.] *W. Wasow.*

Fortet, R. Random determinants. J. Research Nat. Bur. Standards **47**, 465–470 (1951).

The author studies determinants whose elements are random variables X_{ij}. The fundamental assumption made of these random variables is: if $A_1, A_2, \cdots, A_k$ are k random variables chosen from the X_{ij}'s then $E\{X_{ij} \mid A_1, \cdots, A_k\} = 0$ if X_{ij} is not one of the A's. To compare this with the martingale property studied by Doob [Proc. 2d Berkeley Symposium Math. Statist. Prob., 1950, pp. 269–277 (1951); these Rev. **13**, 474] note that the partial sums of any sequence of different X_{ij}'s form a martingale. Expressions for the first and second moments of the determinant are derived and application made to find under certain assumptions the limiting probability of the event "a point in n-dimensional space lies above a random hyperplane." The random variable $A_n h - B$ is considered, where A_n and B are certain minors of the determinantal equation of the random hyperplane and h is a coordinate of the given point. An asymptotic expression for $E\{A_n^2\}$ is obtained, and it is shown that $E\{B^2\}$ is of the order $(1/n)E\{A_n^2\}$.

J. L. Snell (Princeton, N. J.).

Loève, Michel. On almost sure convergence. Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 279–303. University of California Press, Berkeley and Los Angeles, 1951. $11.00.

This is a systematic but highly condensed examination of hypotheses that imply and methods that serve to prove the convergence with probability one of sequences of random variables. It turns out that the hypotheses (of the author's generalizations of classical theorems) are usually quite complicated; the methods, however (being mainly set-theoretic in flavor), are simple and powerful. After an introductory section on classical epsilontic combinatorics, the author studies the concepts of truncation and centering, and, in the last section, the concept of determining set function. (The latter is intimately related to the known concept of conditional expectation with respect to a subalgebra of the algebra of all measurable sets.) He obtains refinements of the known results on convergence of series of random variables, the zero-one law, and the various strong laws of large numbers and ergodic theorems.

P. R. Halmos (Chicago, Ill.).

Dugué, Daniel. Sur la convergence presque certaine au sens de Cesàro de variables aléatoires et sur certaines inégalités concernant les fonctions caractéristiques. C. R. Acad. Sci. Paris **234**, 1837–1840 (1952).

The necessity of the existence of the mean for Kolmogorov's strong law of large numbers for independent, identically distributed random variables is proved. This must be regarded as well-known [see an article by the reviewer, Proc. 2d Berkeley Symposium Math. Statist. and Prob., 1950, Univ. Calif. Press, 1951, pp. 341–352, especially footnote 1, p. 341; these Rev. **13**, 567]. Another theorem gives a necessary and sufficient condition for the existence of the mean in terms of the characteristic function, by means of familiar inequalities. *K. L. Chung* (Ithaca, N. Y.).

Smirnov, N. V. Limit distributions for the terms of a variational series. Amer. Math. Soc. Translation no. **67**, 64 pp. (1952).

Translated from Trudy Mat. Inst. Steklov. **25** (1949); these Rev. **11**, 605.

Takahashi, Shigeru. On the central limit theorem. Tôhoku Math. J. (2) **3**, 316–321 (1951).

Let $f(t)$ be measurable with period 1. Using ordinary measure as probability suppose that $S(t)=n^{-1/2}\sum_{k=1}^{n}f(2^kt)$ converges in law to the normal distribution $G(u)$. It is proved that if $g(t)$ is a non-negative measurable function in $[0,1]$ then $\lim_{n\to\infty}P(|S(t)|\leqq g(t))=\int_0^1 dt\int_{-g(t)}^{g(t)}dG(u)$.

K. L. Chung (Ithaca, N. Y.).

Ammeter, Hans. Wahrscheinlichkeitstheoretische Kriterien für die Beurteilung der Güte der Ausgleichung einer Sterbetafel. Mitt. Verein. Schweiz. Versich.-Math. **52**, 19–72 (1952).

Expository article. *H. L. Seal* (New York, N. Y.).

Mathematical Statistics

Cramér, Harald. Series of lectures on mathematical statistics. Trabajos Estadística **2**, 311–349 (1951). (Spanish)

Lectures given in March 1950 before the Departamento de Estadística, Consejo Superior de Investigaciones Científicas.

Cansado, Enrique. An example of a bivariate distribution. Trabajos Estadística **2**, 261–272 (1951). (Spanish. English summary)

Goch, D. Charleston. The standard error of a ratio. South African J. Sci. **48**, 321–323 (1952).

Moriguti, Sigeiti. A lower bound for a probability moment of any absolutely continuous distribution with finite variance. Ann. Math. Statistics **23**, 286–289 (1952).

Define the nth probability moment of a population with probability density function $f(x)$ by $\Omega_n=\int_{-\infty}^{\infty}[f(x)]^n dx$. It is assumed that the variance σ^2 of $f(x)$ is finite. The author shows that

$$\Omega_n\geqq\frac{2n}{3n-1}\left[\sqrt{\frac{n-1}{3n-1}}\Big/B\left(\frac{n}{n-1},\frac{1}{2}\right)\right]^{n-1}\frac{1}{\sigma^{n-1}},\quad n>1.$$

L. A. Aroian (Culver City, Calif.).

Hess, F. G. Some directional correlation functions for successive nuclear radiations. Canadian J. Physics **30**, 130–146 (1952).

The author reduces the determination of the functions in the title to what is effectively the determination of the ordinary moments of the hypergeometric distribution, for which he gives known results. The Stirling numbers of the second kind, which inevitably appear, are disguised as $a_{k,}$ and tabled for $k=1(1)10$. *J. Riordan.*

Thomson, George W. Measures of plant aggregation based on contagious distributions. Contributions from the Laboratory of Vertebrate Biology, University of Michigan, Ann Arbor, Mich., no. **53**, 17 pp. (1952).

Two types of distribution suggested by J. Neyman [Ann. Math. Statistics **10**, 35–57 (1939)] and M. Thomas [Biometrika **36**, 18–25 (1949); these Rev. **11**, 528], respectively, for the distribution of organisms in spatial flocks are studied in connection with three sets of data. Tables for computing each of the two types of distribution are derived and given.

L. J. Savage (Chicago, Ill.).

Féron, Robert. De la régression. C. R. Acad. Sci. Paris **234**, 2143–2145 (1952).

Stevens, W. L. Mean and variance of an entry in a contingency table. Biometrika **38**, 468–470 (1951).

Patnaik's approximations to the mean and variance [Biometrika **35**, 157–175 (1948), p. 162; these Rev. **9**, 603] are shown to be unsatisfactory under certain conditions not explicitly excluded in the derivation of the approximations. The author gives asymptotic expressions for the mean and variance. *D. F. Votaw, Jr.* (New Haven, Conn.).

Donsker, Monroe D. Justification and extension of Doob's heuristic approach to the Kolmogorov-Smirnov theorems. Ann. Math. Statistics **23**, 277–281 (1952).

The author gives a justification of the heuristic approach to the Kolmogorov-Smirnov theorems given by J. L. Doob [Ann. Math. Statistics **20**, 393–403 (1949); these Rev. **11**, 43]. *J. L. Snell* (Princeton, N. J.).

Cavé, René. Perfectionnement des méthodes modernes de contrôle statistique par mesures. C. R. Acad. Sci. Paris **234**, 2145–2146 (1952).

Steinhaus, H. Quality control by sampling (a plea for Bayes' rule). Colloquium Math. **2**, 98–108 (1951).

The author examines briefly some examples of modern statistical methods of quality control and their refinements by sequential analysis, and gives two accounts of them: in terms of the Bayes' notion of a priori probability, and in terms of the present fashion of repudiating such a notion. He shows some superiorities of the former approach, and emphasizes his view that the second avoids Bayes' notion in appearance only. He concludes by attempting an answer to a well-known objection to the assumption of uniform a priori distribution of the parameter α to be estimated (the objection, namely, that α^2 or some other function β of α will then not be uniformly distributed). His answer is that for quality control α is just one thing, the proportion of good items in the population; this is the variable which (apparently by convention) must be taken as uniformly distributed, not β. *B. O. Koopman* (New York, N. Y.).

Girshick, M. A., and Rubin, Herman. A Bayes approach to a quality control model. Ann. Math. Statistics **23**, 114–125 (1952).

The paper deals with a class of statistical quality control procedures and continuous inspection plans which are optimum for a specified income function and a production model which can be in one of four states, two of which are states of repair, with known transition probabilities. In both a) the case of 100% inspection and b) the case of a continuous inspection plan where inspection costs are considered or the tests are destructive, the Markov process, generated by the model and the class of decision procedures involved, is shown to approach a limiting distribution. Integral equations are obtained in each case from which the optimum procedures can be derived. *R. P. Peterson.*

Cox, D. R. Sequential tests for composite hypotheses. Proc. Cambridge Philos. Soc. **48**, 290–299 (1952).

Let $x=\{x_1, \cdots, x_n\}$ be random variables whose probability density function depends upon p unknown parameters $\theta_1, \cdots, \theta_p$ and suppose that we wish to choose between the hypothesis H_0: $\theta_1=\theta_1^0$ and H_1: $\theta_1=\theta_1^1$ with preassigned probabilities of error α and β using a test which is independent of the nuisance parameters $\theta_2, \cdots, \theta_p$. Suppose that a set of functionally independent jointly sufficient estimators for $\theta_1, \cdots, \theta_p$, say $t_1, \cdots, t_p$ exists for all n such that one of the t_i's, denoted by t_n, has a known density function $g_n(t_n|\theta_1)$ not involving $\theta_2, \cdots, \theta_p$. Further, suppose that the joint probability density function of the t_i's is factorable into the form $g_n(t_n|\theta_1)l_n(t_1, \cdots, t_n)$, where l_n is independent of θ_1, and let $L_n=g_n(t_n|\theta_1^1)/g_n(t_n|\theta_1^0)$. Then the sequential ratio test is defined in the usual way using L_n as the probability ratio. The author gives sufficient conditions for the probability that the test terminates to be one and derives tests for several well known cases. The tests obtained by the author's method are shown to be closely related to those derived by Wald's method using certain weight functions. *R. P. Peterson* (Seattle, Wash.).

Peterson, Raymond P. Uniformly best constant risk and minimax point estimates. J. Research Nat. Bur. Standards **48**, 49–53 (1952).

The main theorem: if the Bayes estimate with respect to Lebesgue measure has constant risk, it is minimax provided either (A) the parameter space is bounded, or (B) the risk integral and the integral of loss with respect to Lebesgue measure are uniformly convergent. Applications to location and scale problems with invariant loss functions. *J. L. Hodges, Jr.* (Chicago, Ill.).

Kitagawa, Tosio. Successive process of statistical inferences. I. Mem. Fac. Sci. Kyūsyū Univ. A. **5**, 139–180 (1950).

The author treats several problems that are very important. In part 1 he treats some problems of estimation where information is pooled from two samples if a test of significance accepts a hypothesis relating the two populations and where information from only the first sample is used otherwise. In part 2 the author treats the problem of using a sample to obtain a confidence type region in which is expected to fall a statistic based on future observations. In this development the attention is confined to problems where the distributions are normal or where at least one of the samples is very large. In part 3 is considered the procedure of successively pooling data from samples until a hypothesis relating the populations from which the samples are drawn is rejected. Then the procedure starts over again. These notions and results are applicable to problems in quality control. *H. Chernoff* (Stanford University, Calif.).

Ottestad, Per. On the test of the hypothesis that the probability of an event is contained within given limits. Skand. Aktuarietidskr. **34**, 197–201 (1951).

Consider a distribution which is the average of binomial distributions, where the parameter p is uniformly distributed over some interval. The author considers the hypotheses $c_1 \leqq p \leqq c_2$ and $b_1 \leqq p \leqq b_2$, where $c_2 \leqq b_1$, and the critical regions are to have selected probabilities α and β. The critical regions are determined by applying the Neyman-Pearson lemma. Despite the author's misgivings the results hold in the general case, even when $c_1 \neq 0$ or $b_2 \neq 1$. In the numerical example to show otherwise the critical region for $b_1 \leqq p \leqq b_2$ should be $x=0$ for $\beta=.06$ rather than $4 \leqq x \leqq 10$. *S. W. Nash* (Vancouver, B. C.).

Lehmann, E. L. A general concept of unbiasedness. Ann. Math. Statistics **22**, 587–592 (1951).

The author defines unbiasedness for decision procedures in the Wald framework of decision theory. According to his definition, the decision procedure $\delta(X)$ is unbiased if, for each θ, $E_\theta W(\theta', \delta(X))$ is minimized when $\theta'=\theta$, where $W(\theta, \delta(X))$ is the loss associated with the decision $\delta(X)$ when the true parameter value is θ. Under appropriate specializations this reduces to Neyman's definition for confidence sets, the Neyman-Pearson definition for tests of hypotheses, the David-Neyman expectation definition for point estimation, and others. The author constructs some reasonable-looking examples for which unbiased decision procedures can exist only if very special conditions are satisfied. Connections between invariance and unbiasedness are pointed out; for example if a unique unbiased procedure exists that uniformly minimizes the risk, then it is almost invariant. Under certain restrictions the converse holds. The author supports with an example his remark to the effect that optimum unbiased procedures are not guaranteed to be satisfactory. *G. W. Brown* (Santa Monica, Calif.).

Kullback, S. An application of information theory to multivariate analysis. Ann. Math. Statistics **23**, 88–102 (1952).

This paper applies the measure of divergence and its properties, as derived in a paper by the author and R. A.

Leibler [same Ann. 22, 79–86 (1951); these Rev. 12, 623], to the problem of discriminating between certain multivariate normal populations by means of linear functions. The divergence, $J(1,2)=I(1:2)+I(2:1)$, between two populations π_1 and π_2, is defined in terms of the measure of information, generalized from Shannon and Weaver's definition. The "best" linear function is found by maximizing the divergence $J'(1,2)$ between the alternative distributions of the linear function. Comparison of $J'(1,2)$ with $J(1,2)$ offers a measure of the discriminating efficiency of the linear function. The classical results on the multivariate analysis techniques of discriminant analysis, principal components, and canonical correlations, corresponding to appropriate assumptions about the underlying populations, are derived from the common background of the divergence, and illustrated with numerical examples taken from the literature. The author's belief that the approach has pedagogical interest is shared by the reviewer. *G. W. Brown.*

Pinel, Jacques. Sur la définition et sur quelques moyens de calcul de l'efficacité d'une méthode de classification et de recherche. C. R. Acad. Sci. Paris 234, 2141–2143 (1952).

Kitagawa, Tosio. Analysis of variance applied to function spaces. Mem. Fac. Sci. Kyūsyū Univ. A. 6, 41–53 (1951).

In this paper the author generalizes the ordinary analysis of variance for a rectangular lay-out (which could well be thought of as a two-way finite stochastic process) to the analogous thing for a certain function space (which may be thought of as a double stochastic process, being a two-parameter family of random variables). The choice of function space and measure is suggested strongly by the assumption of normally distributed random variables used in the ordinary finite case. For this purpose the author employs Wiener measure and the Wiener integral and some properties thereof which have been studied by Cameron, Martin and others. However the above mentioned authors study measure and integral in the space C of continuous functions on $[0,1]$ which vanish at the origin, whereas to have a space which serves as an analogy to a two-dimensional scheme as in the analysis of variance, it is necessary for the author to consider what he calls the space C_2 of continuous functions on $[0,1]\times[0,1]$ vanishing on the lower and left-hand edges. He sketchily indicates a definition of measure and hence integral on this space, and announces that many results that he needs for this integral can be obtained in counter-part from the studies of Cameron and Martin.

He then introduces the functional analogue of row-effects and column-effects, and defines a functional on C_2 which in form preserves the properties possessed by random variables in the finite scheme, and in terms of which the familiar algebraic trick of breaking up variances into sums of sequences persists. Certain functionals related to this basic functional are shown to be normally distributed and independent (in the sense of Wiener measure) on the basis of a calculation founded upon a generalization in C_2 of the translation theorem in C. Having these functionals independent and normally distributed the author proceeds to define functionals in terms of them which are χ^2 distributed, and then sets forth an analysis of variance table in terms of certain hypotheses about the generalized row and column effects. His Theorem 3 parallels the standard analysis of variance scheme for testing these hypotheses. The author also indicates a generalization to the space C_n of functions continuous on the unit n-cube, and indicates the possibility of using his previous methods to establish a generalization of the ordinary Graeco-Latin and hyper-Graeco-Latin squares. *R. E. Fagen* and *R. H. Cameron.*

Deemer, Walter L., and Olkin, Ingram. The Jacobians of certain matrix transformations useful in multivariate analysis. Based on lectures of P. L. Hsu at the University of North Carolina, 1947. Biometrika 38, 345–367 (1951).

This paper presents an effective general attack on the problem of computing Jacobians of matrix transformations. The Jacobians of certain special transformations are evaluated. These transformations and their Jacobians are important in multivariate analysis. In particular, use is made of the fact that non-linear transformations give rise to linear transformations in the differentials. These latter transformations are relevant in the computations of the Jacobians and they are frequently much easier to handle. The presentation of Lemma 4.3 is unclear. *H. Chernoff.*

Murteira, Bento. Note on the variate differences of autoregressive series. Biometrika 38, 479–480 (1951).

Consider a time series x_t generated by the autoregressive scheme

$$x_t+\alpha_1 x_{t-1}+\cdots+\alpha_m x_{t-m}=\epsilon_t+\beta_1\epsilon_{t-1}+\cdots+\beta_n\epsilon_{t-n}.$$

The author finds for the variance of the kth difference of x_t

$$\lim_{k\to\infty}\left[\operatorname{var}(\Delta^k x_t)\Big/\binom{2k}{k}\right]=(\operatorname{var}\epsilon)\left[\frac{1-\beta_1+\beta_2-\cdots+(-1)^n\beta_n}{1-\alpha_1+\alpha_2-\cdots+(-1)^m\alpha_m}\right]^2.$$

A. Blake (Buffalo, N. Y.).

Mathematical Economics

Krentel, W. D., McKinsey, J. C. C., and Quine, W. V. A simplification of games in extensive form. Duke Math. J. 18, 885–900 (1951).

The problem of reducing a two-person zero-sum game in extensive form to the normal form is often a practical impossibility due to the large number of pure strategies. However it is frequently possible to replace the given game by a game with the same value but a reasonable number of pure strategies by the elimination of redundant strategies. This paper gives a method for certain classes of games in extensive form which replaces the given game by an equivalent game with the smallest number of pure strategies. The precise notion of equivalence and description of the reduction process is too lengthy to be described here. *D. Gale.*

***Kuhn, H. W., and Tucker, A. W. Nonlinear programming.** Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 481–492. University of California Press, Berkeley and Los Angeles, 1951. $11.00.

The paper treats the following two related problems. Maximum Problem. Let P_n denote the set of all vectors x in real n-space all of whose components are non-negative. Let F be a differentiable function from P_n into m-space and let g be a real-valued function on P_n. Does there exist a

vector $x \varepsilon P_n$, maximizing g subject to the constraint $F(x) \geqq 0$? Saddle Value Problem. Let φ be a differentiable function defined on $P_n \times P_m$. Do there exist vectors $x_0 \varepsilon P_n$, $u_0 \varepsilon P_m$ such that $\varphi(x, u_0) \leqq \varphi(x_0, u_0) \leqq \varphi(x_0, u)$ for all $(x, u) \varepsilon P_n \times P_m$. The main result of the paper states that if in the maximum problem the function g and each coordinate function of F are concave, then x_0 solves the problem if and only if x_0 and some $u_0 \varepsilon P_m$ solve the saddle value problem for the function φ defined by $\varphi(x, u) = g(x) + \varphi(u, F(x))$. The result is applied to cases where the functions g and F are quadratic and linear functions, and a generalization is given for the case when g is a vector-valued function. *D. Gale.*

Dvoretzky, A., Kiefer, J., and Wolfowitz, J. The inventory problem. I. Case of known distributions of demand. Econometrica **20**, 187–222 (1952).

An ordering policy is a function which associates with each initial stock (more generally, with each past history of the enterprise) an amount to be ordered of the inventoried commodity. An ordering policy is optimal if for every past history it minimizes the expected loss. The authors solve the problem, under very general conditions, of finding an optimal ordering policy, or of constructing one arbitrarily close to optimum, for the case that the probability law of future demand is known. The case of unknown distribution of demand will be treated subsequently. The results generalize those of Arrow, Harris, and Marschak [Econometrica **19**, 250–272 (1951); these Rev. **13**, 368], who consider only a restricted family of ordering policies, here shown not to be broad enough to cover all situations.

Successive sections consider the case of optimization over one time interval, over finitely many, and over an infinite future. In the latter case boundedness considerations require the introduction of a discounting function, as is traditional in problems of this sort. Major attention is given to the important special case where the loss function is the same in each time period and successive demands are independdently distributed. The expected loss for each period is carried back inductively to the initial conditions. From the consideration that an optimal policy must be optimal for the remaining future, no matter what time the process is broken in upon, an integral equation is found for the minimum expected loss as a function of the initial stock, namely:

$$L(x) = \inf_{y \geqq x} \left\{ V(x, y) + \alpha L(0)[1 - F(y)] + \alpha \int_0^y L(y - v) dF(v) \right\}$$

where $L(x)$ is the smallest expected loss, α is the rate of discount, F is the distribution function of demand, and $V(x, y)$ is the expected one-period loss when the initial stock is x and $y - x$ is ordered for immediate delivery. In terms of $L(x)$ the optimal policy itself is characterized. A convergent iterative solution to the integral equation is given, together with bounds on its speed of convergence. This solution is generalized to the case where there is a lag between ordering and delivery; much of the independence of time disappears. Further extensions are indicated to cases where there are several commodities, several customers, and a distributing as well as ordering policy must be decided upon. *R. Solow* (Cambridge, Mass.).

Wald, Abraham. On a relation between changes in demand and price changes. Econometrica **20**, 304–305 (1952).

This is an excerpt from a letter of Wald to Karl Menger, dated January 1940. One of the sufficient conditions for the existence and uniqueness of the solution to a general equilibrium system given earlier by Wald [Erg. Math. Kolloq. **7**, 1–6 (1936); Econometrica **19**, 368–403 (1951); these Rev. **13**, 370] is the following: if $\sigma_j = f_j(s_1, \cdots, s_n)$ are market demand functions giving prices as functions of quantities, and

$$\Delta\sigma_j = f_j(s_1 + \Delta s_1, \cdots, s_n + \Delta s_n) - f_j(s_1, \cdots, s_n)$$

then $\sum \Delta\sigma_j \Delta s_j < 0$ for any Δs_j not all zero. It is here shown that this condition is implied by the negative definiteness of the matrix $[\partial f_j / \partial s_k]$ and the continuity of the second derivatives $\partial^2 f_j / \partial s_i \partial s_k$. It should be noted that negative definiteness is required of the matrix of uncompensated price-quantity responses, and this condition does not follow from the standard theory of consumers' choice. *R. Solow.*

Şerbănescu, Fl. Quelques propriétés caractéristiques des lois de capitalisation par rapport à l'échéance moyenne. An. Acad. Repub. Pop. Române. Secţ. Şti. Mat. Fiz. Chim. Ser. A. **3**, 263–281 (1950). (Romanian. Russian and French summaries)

Let $c(t)$ be a sum of money invested for a period of t years, and let $f(t)$ be the function of capitalization, so that at the end of the period the sum of money has gown to $c(t)f(t)$. The average maturity X is defined by $\int_0^\infty c(t)f(t)dt = f(X)\int_0^\infty c(t)dt$. The author shows that the most general function of capitalization having the property that the average maturity is independent of all parameters in the capitalization function is $f(t; \lambda, \mu, \cdots) = F(\lambda, \mu, \cdots)\Phi(t) + G(\lambda, \mu, \cdots)$. A search for the functions which make the average maturity depend only on the distribution of capital leads to the two families $f(t) = a + bt$ and $f(t) = 1 + \alpha(\beta^t - 1)$, simple and compound interest. *R. Solow* (Cambridge, Mass.).

Froda, Alexandru. Nombres-indices cumulateurs. An. Acad. Repub. Pop. Române. Secţ. Şti. Mat. Fiz. Chim. Ser. A. **3**, 17–62 (1950). (Romanian. Russian and French summaries)

This is not an index-number problem in the usual sense. Suppose p_k is output in year k, $P_{0n} = \sum_0^n p_k$ is total output over $n+1$ years. The author defines an average rate of increase or "cumulative index" i by $p_0 \sum_0^n i^k = P_{0n}$; it is just that steady rate of geometric increase which would yield the observed initial and total outputs. The author shows that this index has certain of the standard properties of a mean and claims that it is superior to measures obtained by averaging annual rates of growth. The continuous analogue (force of interest) is worked out, and some limits on the index are found which are said to be better than those available from standard financial mathematics. *R. Solow* (Cambridge, Mass.).

TOPOLOGY

Dirac, G. A. Some theorems on abstract graphs. Proc. London Math. Soc. (3) **2**, 69–81 (1952).

A graph is k-chromatic if k is the least integer such that the vertices can be coloured in k colours, no two of the same colour being joined by an edge. A k-chromatic graph is critical if it ceases to be k-chromatic whenever one vertex, with its incident edges, is suppressed. The author finds that any k-chromatic graph has a critical k-chromatic subgraph. The purpose of the paper is to study the k-chromatic graphs. The chief results are as follows. If $k \geqq 3$ a k-chromatic

graph has either k vertices each pair of which are joined by an edge or has a circuit of $\geqq k+2$ edges. If $0 \leqq n \leqq k-1$ a k-chromatic graph has either $k-n$ vertices each pair of which are joined by an edge or has $\geqq k+n+2$ vertices. If $k \geqq 3$ a critical k-chromatic graph of $\leqq 2k-1$ vertices has a circuit which passes through every vertex. In the last result the author conjectures that $2k-1$ can be replaced by Bk^2 for some positive constant B.

The paper begins with a series of auxiliary theorems which are of interest in themselves. Each is concerned with the length of the longest circuit existing in a graph satisfying certain conditions. For example, it is shown that if $L(n, d)$ is the least integer for which there exists a non-separable graph of n vertices, each of degree $\leqq d$, whose longest circuit has length $L(n, d)$, then $L(n, d)$ tends to infinity with n.

W. T. Tutte (Toronto, Ont.).

Dirac, G. A. The colouring of maps. Nature **169**, 664 (1952).

This announces some results on the colouring of maps whose proofs are to appear elsewhere. They give a method for determining k_h, the maximum integer n such that some map on a surface of connectivity h requires n colours, for all $h>1$. (The 4-colour problem is concerned with the case $h=1$.) *W. T. Tutte* (Toronto, Ont.).

Frucht, Robert. A one-regular graph of degree three. Canadian J. Math. **4**, 240–247 (1952).

An s-arc in a cubic graph is any path $A_0A_1 \cdots A_s$ formed by s consecutive arcs $A_0A_1, A_1A_2, \cdots, A_{s-1}A_s$ and taken in a definite sense. The graph is s-regular if it is connected and for any two s-arcs S_1 and S_2 there is a unique element of the group of the graph which transforms S_1 into S_2. Such graphs have been studied by the reviewer [Proc. Cambridge Philos. Soc. **43**, 459–474 (1948); these Rev. **9**, 97] and by Coxeter [Bull. Amer. Math. Soc. **56**, 413–455 (1950); these Rev. **12**, 350]. It has been shown that the case $s \geqq 6$ is impossible, and examples have been given for the cases $s=2, 3, 4$ and 5.

In this paper the author constructs the first example of a 1-regular cubic graph. It has 432 vertices.

W. T. Tutte (Toronto, Ont.).

van Aardenne-Ehrenfest, T., and de Bruijn, N. G. Circuits and trees in oriented linear graphs. Simon Stevin **28**, 203–217 (1951).

In an "oriented" graph each edge is regarded as directed from one end, called its tail, to the other, called its head. In the case of a loop the head and tail coincide. The authors define a T-graph as an oriented graph in which each vertex is the head and tail of equal numbers of edges. A circuit in a T-graph is a unicursal path in the graph whose direction of description is always from the tail of an edge to the head. By a tree with root A in an oriented graph G the authors mean a tree in the ordinary sense having all the vertices of G as its vertices and a subset of the edges of G as its edges, with the proviso that each edge of the tree is directed away from A in the tree. The authors give a new proof of a theorem of the reviewer which expresses the number of trees in G with a given root as a determinant. They remark that this number is independent of the root in the case of a T-graph.

The authors prove a simple formula giving $|T|$, the number of circuits of a T-graph T, in terms of the number of trees with a given root. From T other T-graphs may be constructed. One, T^λ, is formed by replacing each edge by λ similarly directed edges. Another, T^*, exists when all the vertices of T have the same degree. Its vertices are the edges of T. Two of its vertices e_i and e_j are joined by a single edge with tail e_i if the tail of e_j is the head of e_i in T. Otherwise they are not joined by an edge with tail e_i. Among other interesting theorems the authors give formulae for $|T^\lambda|$ and $|T^*|$ in terms of $|T|$. The theory depends on some simple properties of the symmetric group which are proved in the early part of the paper. *W. T. Tutte.*

Motchane, Léon. Espaces compacts au sens de la convergence simple. C. R. Acad. Sci. Paris **233**, 1569–1571 (1951).

$\mathfrak{F}$ désigne une famille d'applications f d'un ensemble abstrait E dans un espace K métrique compact. Une suite d'applications de $\mathfrak{F}$ est regardée comme convergente (convergence simple) si elle converge ponctuellement. Une suite de familles $\mathfrak{E}_n$ d'ensembles de E forme un "système essentiel $\mathbf{E}$" si pour tout n, il existe une sous-famille dénombrable $\mathfrak{D}_n = \{E(n, 1), \cdots, E(n, i), \cdots\}$ de $\mathfrak{E}_n$ telle que l'union des ensembles de $\mathfrak{E}_n$ rencontrant $s(n) = \bigcup_i E(n, i)$ soit $= E$. $\mathfrak{F}$ est "équi-associée" à $\mathbf{E}$ si l'oscillation de tout f sur tout ensemble de $\mathfrak{E}_n$ est $< 1/n$. Théorème 1: Si $\mathfrak{F}$ est équi-associée à un système essentiel $\mathbf{E}$, l'espace $\mathfrak{F}$ est compact et métrisable. Idée de la démonstration: Extrayant de chaque $E(n, i)$ un point, on définit un ensemble dénombrable D tel que la convergence simple des f sur D entraîne la convergence sur E. Théorème 2: Si l'espace $\mathfrak{F}$ est compact et métrisable, la famille $\mathfrak{F}$ est équi-associée à un système essentiel $\mathbf{E}$. Ces théorèmes constituent la synthèse annoncée dans une note antérieure [mêmes C. R. **231**, 1206–1208 (1950); ces Rev. **13**, 21], comprenant comme cas particuliers les familles (1) également continues, (2) équi-quasicontinues, (3) également continues en mesure, (4) également quasicontinues en mesure. [Remarques. (i) Le rapporteur comprend, dans Théorème 1, "compact" au sens de "précompact", aucune hypothèse n'ayant été faite assurant la présence des fonctions limites dans $\mathfrak{F}$. (ii) Les exemples (3) et (4) se déduisent de (1) et (2) remplaçant la topologie euclidienne par la topologie D concrète introduite par Haupt et Pauc [ibid. **234**, 390–392 (1952); ces Rev. **13**, 728].] *C. Pauc* (le Cap).

Grothendieck, A. Critères de compacité dans les espaces fonctionnels généraux. Amer. J. Math. **74**, 168–186 (1952).

(In this review, all spaces are Hausdorff spaces and "if and only if" is rendered by "iff".) For a subset A of a space H, define certain statements as follows: $\rho\alpha A$ [αA]: A is relatively compact [compact] iff every ultra-filter on A converges to a point of H [of A] (or equivalently, iff every open covering of A [A] admits a finite subcovering). $\rho\beta A$ [βA]: A is relatively semi-compact [semi-compact] iff every sequence in A has a cluster point in H [in A]. $\rho\gamma A$ [γA]: A is relatively strictly semi-compact [strictly semi-compact] iff every sequence in A has a subsequence which converges to a point of H [of A]. The author obtains conditions under which certain of these statements are equivalent. We describe only the results for $\rho\alpha$, $\rho\beta$, and $\rho\gamma$.

It has long been known that $\rho\alpha \rightarrow \rho\beta$, $\rho\gamma \rightarrow \rho\beta$, $\rho\alpha \leftrightarrow \rho\gamma$, $\rho\gamma \leftrightarrow \rho\alpha$, and that in a (i.e., for all subsets of a) metrisable space, $\rho\alpha \leftrightarrow \rho\beta \leftrightarrow \rho\gamma$. More recently, it has been shown by Weil [Sur les espaces à structure uniforme et sur la topologie générale, Actualités Sci. Ind., no. 551, Hermann, Paris, 1937] that $\rho\alpha \leftrightarrow \rho\beta$ in every complete uniform space, and by Šmulian and Eberlein [Proc. Nat. Acad. Sci. U. S. A. **33**, 51–53 (1947); these Rev. **9**, 42; **10**, 855] that $\rho\alpha \leftrightarrow \rho\beta \leftrightarrow \rho\gamma$ in every Banach space with its weak topology. These results form the author's point of departure.

Using in part a technique of Eberlein, he proves (1): Suppose 'E is semi-compact, F is a uniform space, S is a covering of E, and $C_S(E, F)$ is the space of continuous functions on E to F, topologized by uniform convergence on the sets of S.' Then $\rho\beta \leftrightarrow \rho\alpha$ in $C_S(E, F)$ if $\rho\beta \leftrightarrow \rho\alpha$ in F (in particular, if F is complete). Observing that (1) applies to closed subsets of $C_S(E, F)$, that E need merely be such that a function on E to F is continuous iff it is continuous on every semi-compact subset of E, and that a locally convex linear space L can be regarded as a closed subset of $C_T(L, L')$, where L' is the weak dual of L and T a certain family of weakly compact subsets of L', he applies (1) to prove (1_L): In a complete locally convex linear space with the weak topology, $\rho\beta \leftrightarrow \rho\alpha$. Concerning $\rho\gamma$ he proves (2): Suppose '$\cdots$' of (1) holds, $A \subset C_S(E, F)$, and $f(E)$ is metrisable for each $f \in A$. Then $\rho\beta A \leftrightarrow \rho\gamma A$. Applied to linear spaces, (2) yields (2_L): Suppose $x_1, x_2, \cdots$, are points of the locally convex linear space L such that $\{x_1, x_2, \cdots\}$ is weakly relatively semi-compact. Then for each weakly compact set $K \subset L'$, x_n has a subsequence y_n such that $f(y_n)$ converges for each $f \in K$. If in L the neutral element 0 admits a sequence of neighborhoods whose intersection is $\{0\}$, then $\rho\beta \leftrightarrow \rho\gamma$ in L with its weak topology.

The paper contains several interesting applications of these results, such as further characterizations of weak compactness in general locally convex linear spaces and in the space of continuous complex-valued functions on a topological space. An application to weakly almost-periodic functions is given, and the following result is stated: If L is a locally convex linear space, A a weakly relatively compact subset of L, and C the closed convex hull of A, then C is weakly compact iff C is complete. *V. L. Klee, Jr.*

Menger, Karl. A topological characterization of the length of paths. Proc. Nat. Acad. Sci. U. S. A. **38**, 66–69 (1952).

T denotes a compact metrizable topological space. For any metrization of T, the length l of paths (directed curves) $\mathfrak{P}$ is (A) additive: if $\mathfrak{Q}$ is consecutive to $\mathfrak{P}$, then

$$l(\mathfrak{P}+\mathfrak{Q}) = l(\mathfrak{P}) + l(\mathfrak{Q});$$

(B) lower semi-continuous; (C) regular: if, for each n, $\mathfrak{P}_n$ is a path from p to p_n and $\lim l(\mathfrak{P}_n) = 0$, then $\lim p_n = p$. If λ is any non-negative, finite or infinite function defined on all paths of T, having properties (A), (B) and (C), then, for every $L < \infty$, the set of all paths $\mathfrak{P}$ for which $\lambda(\mathfrak{P}) \leqq L$, is compact (Generalized Carathéodory Theorem). By metroid is meant a set in which every two points have a finite or infinite "distance" such that $d(p, p) = 0$, $d(p, q) > 0$ if $p \neq q$, $d(p, q) + d(q, r) \geqq d(p, r)$. Main Theorem: In order that λ be the length in a metroid M which (1) contains a path of length $d(p, q)$ from p to q whenever $d(p, q) < \infty$, (2) can be mapped on T in a one-to-one continuous way and so that $l(\mathfrak{P}) = \lambda(\mathfrak{P})$ for any path, it is necessary and sufficient that λ have the properties (A), (B), (C). Complements: If T is λ-connected (i.e., contains, for any two points p, q a path $\mathfrak{P}$ for which $\lambda(\mathfrak{P}) < \infty$), then the M-distances can be chosen to be finite. If T is locally λ-connected (i.e., contains, if $\lim p_n = p$, a path $\mathfrak{P}_n$ from p to p_n such that $\lim \lambda(P_n) = 0$), then there exists an M which can be mapped on T topologically. Hint to the proof for sufficiency and complements: Assuming that λ has properties (A), (B), (C), the function $d(p, q) = \liminf \lambda(\mathfrak{P})$ for all paths $\mathfrak{P}$ from p to q, if p and q are path-connected, $= \infty$ if there is no path joining p to q, makes T a metroid T^*. (B) secures the identity of the d-length l and λ. The existence of the path (segment) required in (1) follows from the definition of d, property (C) and the generalized Carathéodory theorem. If we define (see remark below) the convergence of a sequence $p_1, \cdots, p_n, \cdots$ to p in T^* by $\lim d(p, p_n) = 0$, then the identical (or canonical) mapping $x \leftrightarrow x$ of T^* on T is continuous on account of (C). If T is λ-connected, $d(p, q)$ is finite. If T is locally λ-connected, then $\lim p_n = p$ in T implies $\lim d(p, p_n) = 0$, in other words $\lim p_n = p$ in T^*. [Remark. There is no indication in the note as to the definition of convergence adopted in a metroid. The conjecture above is interpretation of the reviewer. If it is correct, the proof for necessity must involve a new technique, namely to establish the lower semi-continuity of the length in the metroid M satisfying (1) and (2). The classical proof rests on the property for $d(p, p_n)$ and $d(p_n, p)$ to tend to zero if $p_n \to p$.] *C. Y. Pauc* (Cape Town).

Dolcher, Mario. Geometria delle trasformazioni continue. Un rafforzamento di enunciati precedenti. Rivista Mat. Univ. Parma **2**, 331–335 (1951).

Let T be a continuous mapping from a simply connected region R, lying in a plane p, into a plane p', and let C be the boundary curve of R. In p', let Q_0 be a point with the following properties. (a) Q_0 non-ε $T(C)$. (b) The topological index O of Q_0 with respect to $T(C)$ satisfies the inequality $O \geqq n > 0$, where n is an integer. In connection with studies on surface area and on generalized Jacobians, several results have been obtained in the literature to the general effect that as a consequence of the assumptions stated, a certain vicinity of the point Q_0 will be covered at least n times by $T(R)$. In the present note, the author improves previous results of this type. *T. Rado* (Columbus, Ohio).

Trevisan, Giorgio. Punti uniti in trasformazioni del cerchio. Giorn. Mat. Battaglini (4) **3**(**79**), 127–131 (1950).

Let G be a circular disk in E_2 and t a continuous mapping $G \to E_2$ such that t maps the circle g, boundary of G, topologically. Assume that $t(g)$ contains interior points of G. The author obtains sufficient conditions for the existence of fixed points of t in terms of the intersectional properties of certain subarcs of $g \cup t(g)$. Recent papers on this subject: G. Scorza Dragoni, Ann. Mat. Pura Appl. (4) **25**, 43–65 (1946); Mario Volpato, ibid. **27**, 101–105 (1948); these Rev. **9**, 455; **10**, 728. *P. A. Smith* (New York, N. Y.).

Fan, Ky. Fixed-point and minimax theorems in locally convex topological linear spaces. Proc. Nat. Acad. Sci. U. S. A. **38**, 121–126 (1952).

Let L be a locally convex linear topological space. The author's key theorem is: If K is a compact convex subset of L and f defines an upper semicontinuous correspondence between points of K and non-empty closed convex subsets of K, then for some $x_0 \in K$, $x_0 \in f(x_0)$. For applications the author gives a general lemma for product spaces. For the case that a product of two spaces are involved, a minimax theorem follows immediately which is the generalization of the von Neumann result replacing Euclidean spaces by L spaces and the viewpoint is that introduced by Kakutani for the Euclidean space case. [Reviewer's note: Actually the key theorem, or the essentially equivalent theorem given by I. L. Glicksberg [Proc. Amer. Math. Soc. **3**, 170–174 (1952); these Rev. **13**, 764] and hence also the slightly weaker result published by Bohnenblust and Karlin [Contributions to the theory of games, Princeton Univ. Press, 1950; these Rev. **12**, 844] is easily shown to be a special case of a well known fixed point theorem [cf. Begle, Ann. of Math. (2) **51**, 544–

550 (1950); these Rev. 11, 734]. Accordingly, the reviewer can show that considerable generalization of the minimax theorem to non-convex situations is possible.]

D. G. Bourgin (Urbana, Ill.).

✓ ***Leray, Jean. La théorie des points fixes et ses applications en analyse.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 202–208. Amer. Math. Soc., Providence, R. I., 1952.

This is a resumé of a part of fixed point theory essentially described as the result of replacing the finite-dimensional Euclidean space by a locally convex linear topological space. The central notion of degree and index of a mapping is introduced following the pattern of the Leray-Schauder presentation. In the main the results are covered by Leray, J. Math. Pures Appl. (9) **24**, 201–248 (1946); these Rev. **7**, 468.

D. G. Bourgin (Urbana, Ill.).

Vaĭnšteĭn, I. A. On one-dimensional mappings. Doklady Akad. Nauk SSSR (N.S.) **83**, 175–178 (1952). (Russian)

Let X and Y be metric separable spaces, and f a continuous and closed map of X onto Y. The map f is called k-dimensional if $\dim f^{-1}(y) \leqq k$ for every y of Y. Let $\mu(y)$ denote the power of the set $f^{-1}(y)$, $y \in Y$, let $Y_k = \{y; \mu(y) \geqq k\}$ and $Y_k^0 = \{y; \dim f^{-1}(y) \geqq k\}$. If A is a subset of X, and f is restricted to A, then the corresponding sets are denoted by $Y_k(A)$ and $Y_k^0(A)$. For zero-dimensional maps with $\dim Y = \dim X + n$, there is a formula of Freudenthal: 1) $\dim Y_k \geqq \dim Y - k + 1$ $(k = 1, \cdots, n+1)$. This formula does not hold for higher-dimensional maps. The present note discusses formulas valid for one-dimensional maps, and states corresponding formulas for the case of higher-dimensional maps, as follows. Lemma 3: Suppose that f: $X \to Y$ is a one-dimensional map; then 2) $\dim Y \leqq n$, and $\dim Y_1^0 \leqq n-1$ implies that $\dim X \leqq n$. Lemma 4: Suppose that f: $X \to Y$ is a one-dimensional map; then 3) $\dim X \leqq n$, $\dim Y_2 \leqq n-1$, $\dim Y_1^0 \leqq n-2$ implies that $\dim Y \leqq n$. Lemma 5: Suppose that f: $X \to Y$ is a one-dimensional map and $\dim Y_k > \dim X$; then $\dim Y_{k+1} \geqq \dim Y_k - 2$ and if $\dim Y_{k+1} = \dim Y_k - 2$ then $\dim Y_1^0 = \dim Y_k - 2$.

Theorem 1. Suppose that f: $X \to Y$ is a one-dimensional map and $\dim Y = \dim X + n$; suppose, further, that $\dim Y_1^0 = \dim Y - m$. Then for $k \leqq n+1$ it is true that 1) if $k < m$, then $\dim Y_k \geqq \dim Y - k + 1$, and 2) $\dim Y_m \geqq \dim Y - m$. For k-dimensional maps, the analogous formulas read: 4) $\dim Y \leqq n$, $\dim Y_1^0 \leqq n-1$, $\cdots$, $\dim Y_k^0 \leqq n-k$ implies that $\dim X \leqq n$, and 5) $\dim X \leqq n$, $\dim Y_2 \leqq n-1$, $\dim Y_1^0 \leqq n-2$, $\cdots$, $\dim Y_k^0 \leqq n-(k+1)$ implies that $\dim Y \leqq n$.

L. Zippin (Flushing, N. Y.).

Gordon, I. I. Classification of the mappings of an n-dimensional complex into an n-dimensional real projective space. Izvestiya Akad. Nauk SSSR. Ser. Mat. **16**, 113–146 (1952). (Russian)

Detailed exposition of results reported on in an earlier note [Doklady Akad. Nauk SSSR (N.S.) **65**, 441–444 (1949); these Rev. **10**, 617]. The author remarks that M. M. Postnikov independently obtained the solution to a more general problem, using more involved methods [ibid. **66**, 161–164; **67**, 427–430 (1949); these Rev. **11**, 451].

L. Zippin (Flushing, N. Y.).

Shimada, Nobuo, and Uehara, Hiroshi. Classification of mappings of an $(n+2)$-complex into an $(n-1)$-connected space with vanishing $(n+1)$-st homotopy group. Nagoya Math. J. **4**, 43–50 (1952).

This paper is concerned with the homotopy classification of mappings of an $(n+2)$-dimensional complex into an arcwise connected space Y whose homotopy groups $\pi_i(Y)$ vanish for $i < n$ and for $i = n+1$, and whose n-dimensional homotopy group is finitely generated $(n > 2)$. In order to solve this classification problem, the authors introduce some new cohomology operations. One of these, which they call the $\mathfrak{Z}_r$-square, is a homomorphism of the n-dimensional cohomology group with coefficients integers mod m (m even) into the $(2n-i)$-dimensional integral cohomology group, where $n-i$ is odd. This homomorphism is defined by means of Steenrod's cup-i, cup-$(i+1)$, and cup-$(i+2)$ products, and is discussed rather completely. Some variations of this operation are introduced with prctically no discussion. The methods used to prove the homotopy classification theorem are analogous to those of Steenrod [Ann. of Math. (2) **48**, 290–320 (1947); these Rev. **9**, 154]. As an application of this theorem, they give a formula for expressing the Eilenberg-MacLane invariant $\mathbf{k}^{n+3}$ for the space Y [for the definition of $\mathbf{k}^{n+3}$, see Ann. of Math. **51**, 514–533 (1950); these Rev. **11**, 737] by means of these new operations.

Reviewer's note: The statement of lemma 3, part (β), of this paper is not correct. For the correct statement the reader is referred to lemmas 3.6 and 5.4 of a recent paper by P. Hilton [Quart. J. Math., Oxford Ser. (2) **2**, 228–240 (1951); these Rev. **13**, 267] and a note by M. Barratt and G. Paechter [Proc. Nat. Acad. Sci. U. S. A. **38**, 119–121 (1952); these Rev. **13**, 674]. *W. S. Massey.*

Jackson, James R. Spaces of mappings on topological products with applications to homotopy theory. Proc. Amer. Math. Soc. **3**, 327–333 (1952).

If A and B are arbitrary topological spaces, let B^A denote the compact-open topologized space of continuous maps of A into B. This topology is defined by taking as a sub-base the collection of sets $(K, W) = \{f | f \in B^A, f(K) \subset W\}$ for K a compact subset of A and W an open subset of B. The author first establishes a lemma to the effect that, if A is a Hausdorff space and if $\{U\}$ is a sub-base for the topology in B, then the collection (K, U), for K a compact subset of A and $U \in \{U\}$, is a sub-base for the compact-open topology in B^A. Next, let X and Y be Hausdorff spaces and Z be an arbitrary topological space. The relations between the spaces $Z^{X \times Y}$, Z^X, and $(Z^X)^Y$ are studied. By means of the lemma, the author proves his basic theorem which is stated as follows. For $f \in Z^{X \times Y}$, define $\sigma(f)$: $Y \to Z^X$ by $\sigma(f)(y)(x) = f(x, y)$, $x \in X$, $y \in Y$. Then σ is a homeomorphism of $Z^{X \times Y}$ into $(Z^X)^Y$. Applications to the homotopy groups of function spaces are given at the end of the paper. *S. T. Hu.*

Hu, Sze-Tsen. On products in homotopy groups. Univ. Nac. Tucumán. Revista A. **8**, 107–119 (1951).

Let $A \subset X$. Using the Whitehead pairing $\pi_p(A)$, $\pi_q(A)$ to $\pi_{p+q-1}(A)$, one gets by evident injection, a pairing $\pi_p(A)$, $\pi_q(X)$ to $\pi_{p+q-1}(X)$. In addition, the author defines a pairing $\pi_p(A)$, $\pi_q(X, A)$ to $\pi_{p+q-1}(X, A)$. Using these pairings, it is shown that for fixed $\alpha \in \pi_p(A)$, $\beta \to [\alpha, \beta]$ defines a map of the homotopy sequence of (X, A) in itself with commutativity in every square. By means analogous to those of Steenrod [Ann. of Math. **50**, 954–988 (1949); these Rev. **11**, 122] this leads to a "functional product": if f: $X \to Y$ and $\alpha \in \pi_p(X)$, $\beta \in \pi_q(X)$, have Whitehead product zero, and

also $f^+(\alpha)=f^+(\beta)=0$, then $\alpha \oplus_f \beta \in \pi_{p+q}(Y)/f^+\pi_{p+q}(X)$ is defined and is a homotopy invariant. A few properties of this functional product are listed. *J. Dugundji.*

Hu, Sze-Tsen. A cohomology theory with higher coboundary operators. III. The homotopy axiom and the groups for spheres. Nederl. Akad. Wetensch. Proc. Ser. A. 55 = Indagationes Math. 14, 123–129 (1952).

Completion of the proof that the author's cohomology theory [same Proc. 52, 1144–1150 (1949); 53, 70–76 (1950); these Rev. 11, 452] satisfies all the Eilenberg-Steenrod axioms except the dimension axiom (which it does not). The groups for spheres are calculated. *J. Dugundji.*

Sitnikov, K. The duality law for non-closed sets. Doklady Akad. Nauk SSSR (N.S.) 81, 359–362 (1951). (Russian)

For arbitrary complementary sets A, B in a (homology) manifold M^n, which is acyclic in dimensions q and $q+1$, the author proves a general duality law stating that $\nabla^p A$ and $\Delta^q B$ $(p+q=n-1)$ are isomorphic. Here $\nabla^p A$ and $\Delta^q B$ are (co-)homology groups of A and B, over the same coefficient group, defined as follows: A q-cycle in B is a sequence $\{z_k^q, x_k^{q+1}\}$ of ϵ_k-cycles and chains, $\epsilon_k \to 0$, all lying in some compact subset of B, such that $\partial x_k = z_{k+1} - z_k$. A cycle bounds if there exist $\{y_k^{q+1}, x_k^{q+2}\}$ ϵ'_k-chains with $\epsilon'_k \to 0$, lying in a compact subset of B, such that

$$\partial y_k = z_k, \quad \partial x_k^{q+2} = y_{k+1} - y_k - x_k^{q+1}.$$

Cycles modulo bounding cycles form the homology group $\Delta^q B$; $-\nabla^p A$ is the cohomology group of A, defined as the direct limit of the cohomology groups (infinite cochains) of the nerves of (infinite) star-finite coverings of A. For the proof of the duality law the author uses a third type of groups. Consider triangulated neighborhoods of A; they form a directed system in an obvious way. There are natural maps of the infinite cycles and co-cycles of a triangulated neighborhood to a refining triangulated sub-neighborhood (both maps in the same direction); the direct limits are called $\Delta_\nu^r A$ and $\nabla_\nu^r A$. By Poincaré duality which is compatible with the maps one gets an isomorphism of $\nabla_\nu^p A$ and $\Delta_\nu^{q+1} A$.

The author shows now that (a) $\nabla^p A \approx \nabla_\nu^p A$, and (b) $\Delta_\nu^{q+1} A \approx \Delta^q B$. (a) For each open covering of A one considers the neighborhood formed by the sets of the covering, and triangulates this neighborhood so fine that there is a canonical map into the nerve of the covering. Using the induced cohomology map one easily establishes the isomorphism. (b) For each $(q+1)$-cycle u on a triangulation of a neighborhood of A one constructs an element of $\Delta^q B$ by representing the neighborhood as union of an increasing sequence of finite open sub-complexes t_k, letting z_k^q be the boundary of the part u_k of u in t_k, and putting $x_k^{q+1} = u_{k+1} - u_k$. An "infinitely small" displacement will move all this into the complement of the neighborhood (a compact subset of B). This map is then shown to be an isomorphism; the reasoning is related to Steenrod's theory of regular cycles [Ann. of Math. (2) 41, 833–851 (1940); these Rev. 2, 73]. The connection with Alexandrov's duality law [Mat. Sbornik 21(63), 161–232 (1947); these Rev. 9, 456; note also the related work of S. Kaplan, Trans. Amer. Math. Soc. 62, 248–271 (1947); these Rev. 9, 456], involving the "non-linking" cycles in B and a similar subgroup of the cohomology group of A, is briefly considered; the two subgroups are topological invariants; compact coefficients have to be used. *H. Samelson* (Ann Arbor, Mich.).

Sitnikov, K. On continuous deformations of nonclosed sets. Doklady Akad. Nauk SSSR (N.S.) 82, 845–848 (1952). (Russian)

Theorem: In the n-dimensional closed (locally Euclidean triangulable) manifold M consider a set A and a cycle z (in the sense of the author's paper reviewed above) in a compact set $\Phi \subset M-A$; let g_θ and h_θ, $0 \leqq \theta \leqq 1$, be a deformation of A and Φ such that $g_\theta A$ and $h_\theta \Phi$ are disjoint, and suppose $h_1 z \sim 0$ in $M - g_1 A$; then $z \sim 0$ in $M-A$. Here A is arbitrary (not necessarily closed); this makes the theorem nontrivial. The necessary constructions take place in the product $M \times I$ of M and the unit interval I. In particular one considers the following sets: $A \times I \cup (M-\Phi) \times 0$, and its image under g_θ (i.e., apply g_θ at level θ), and $M \times I - (\Pi \cup u \times 1)$, where Π is the image of $\Phi \times I$ under h_θ, and u is a set carrying the homology $h_1 z \sim 0$ in $M - g_1 A$. The theorem can be extended to open manifolds by assuming that the deformations g and h tend to zero if one approaches "infinity".

H. Samelson (Ann Arbor, Mich.).

Sitnikov, K. On the dimension of non-closed sets of Euclidean space. Doklady Akad. Nauk SSSR (N.S.) 83, 31–34 (1952). (Russian)

The theorem proved gives a solution to a problem posed by P. S. Alexandrov in 1935: Let A be an arbitrary r-dimensional set in Euclidean n-space R^n, $0 \leqq r \leqq n-1$; then (1) every cycle of dimension $< n-r-1$ (arbitrary coefficients) in $R^n - A$, if it bounds in an open set U, bounds already in $U-A$; (2) on the other hand there exist an open cell U in R^n and an $(n-r-1)$-cycle in $U-A$ (integral coefficients) which does not bound in $U-A$. Cycles and homologies are taken in the sense of the author [reference cited in the preceding review]. (R^n can be replaced by any locally Euclidean triangulable manifold.) (1) is easily proved, using a theorem of the author in the paper reviewed above. For (2), the hypothesis implies the existence of a map f of an open set O_A containing A into an r-simplex T^r, which is essential on A; one can assume O_A triangulated and f simplicial. In O_A one has then the obstruction co-cycle against deforming f into the boundary $\dot{T}^r$ of T^r, and the dual cycle z^{n-r}. Assuming (2) not to hold, it is then shown that $z^{n-r} \sim 0$ (in the sense of the author), namely a closed set F in $O_A - A$ can be found such that $z^{n-r} \sim 0$ in $O_A - F$. This is done by removing z^{n-r} from A cell by cell, starting with the vertices: consider a simplex t^{n-r} of z^{n-r} and a convex open cell V containing it. For each vertex t^0 a point Φ^0 in $V-A$, and an infinite 1-drain x^1 are found such that $t^0 \sim 0$ in $V - \Phi^0$, with $\partial x^1 = t^0$. For an edge t^1 with vertices t^0_1, t^0_2 the cycle $\Phi^0_2 - \Phi^0_1$ bounds on a compact set Φ^1 in $V-A$; using this, one shows that the 1-chain $t^1 + x^1_1 - x^1_2$ bounds in $V - \Phi^1$, etc. For the last step one has to assume that (2) fails. It follows then that the obstruction is 0, so that f is inessential on the r-skeleton of O_A, and then also on A; contradiction, q. e. d.

Another approach is sketched, involving a strengthening of the author's duality theorem [see the second preceding review] and a theorem giving a homology characterization of dimension: dim A is the largest r for which there are sets G, H in A ($\bar{H} \subset G$) and an (infinite) covering ω of G, whose nerve contains an r-cocycle which lies actually in H, but doesn't bound there, even if projected into an arbitrary refinement of ω.

H. Samelson (Ann Arbor, Mich.).

Hopf, Heinz. Über komplex-analytische Mannigfaltigkeiten. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **10**, 169–182 (1951).

In this expository paper, the author tells what a complex analytic manifold is, describes some of the fundamental problems that occur in the general theory of complex analytic manifolds, and gives a resumé of the principal results which have been attained so far toward a solution of these problems. The exposition is in a very clear and readable style. *W. S. Massey* (Providence, R. I.).

GEOMETRY

Bruins, E. M. Orthogonal transversals in the tetrahedron. Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 164–172; corrigendum 346 (1952).

Through a point three transversals may be drawn meeting the three pairs of opposite edges of a given tetrahedron. Could the point be chosen so that the three transversals would be mutually orthogonal? The author shows analytically that the problem has, in general, eight solutions. Special cases of the problem are also considered.

N. A. Court (Norman, Okla.).

Marmion, Alphonse. Extension au simplex de l'espace euclidien à n dimensions d'une propriété du triangle et du tétraèdre. C. R. Acad. Sci. Paris **234**, 2040–2042 (1952).

If a point describes a straight line, its pedal sphere (circle) for a tetrahedron (triangle) remains orthogonal to a fixed sphere (circle). The author gives a new proof of this proposition for the case of the tetrahedron which he readily adapts to the n-dimensional case. The proof involves a minimum of computations. *N. A. Court* (Norman, Okla.).

Cavallaro, Vincenzo G. Sur les triangles brosteineriens. Anais Fac. Ci. Porto **36**, 21–25 (1952).

Cavallaro, Vincenzo G. Sur les points isodynamiques. Anais Fac. Ci. Porto **36**, 5–20 (1952).

Obláth, Richard. Ein Beitrag zur Theorie der geometrischen Konstruktionen. Acta Sci. Math. Szeged **14**, 101–102 (1951).

The author proves that all quadratic constructions can be carried out by means of a ruler if an arbitrarily small fixed arc of a circle is given on which the two points trisecting the arc are marked. It is shown that the centre of the circle can be constructed, and the result then follows from a generalization of Steiner's theorem proved independently by F. Severi [Rend. Cir. Mat. Palermo **18**, 256–259 (1904)], the author [Monatsh. Math. Phys. **26**, 295–298 (1915)], and J. Sz. Nagy [Tôhoku Math. J. **40**, 76–78 (1935)].

F. A. Behrend (Melbourne).

Niggli, Alfred, und Niggli, Paul. Raumgruppensymmetrie und Berechnungsmethoden der Kristallstrukturlehre. II. Z. Angew. Math. Physik **2**, 311–336 (1951).

In Fortsetzung des ersten Teiles [dieselbe Z. **2**, 217–232 (1951); diese Rev. **13**, 269] werden Methoden angegeben um die Summationsverfahren zu vereinfachen, die bei der Berechnung des Strukturvektors und der Elektronendichte auftreten. Diese beruhen auf passender Verwendung der Symmetrieeigenschaften des Objektes, die Charakterentafeln ermöglichen ihre Anwendung. *J. J. Burckhardt.*

Amante, Salvatore. Su alcune proprietà delle coniche e quadriche a centro. Ricerca, Napoli **3**, no. 1, 13–19 (1952).

Laurenti, Fernando. Sull'equazione delle tangenti tirate da un punto ad una conica con applicazioni. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **85**, 389–395 (1951).

Giovanardi, Mario. Sull'uso combinato del metodo di Monge e del metodo della proiezione centrale per la determinazione della prospettiva di una superficie di rotazione. Ricerca, Napoli **3**, no. 1, 20–30 (1952).

Jonas, Hans. Deutung einer birationalen Raumtransformation im Bereiche der sphärischen Trigonometrie. Math. Nachr. **6**, 303–314 (1952).

Let $P'(x', y', z')$ be the intersection of the polar planes of $P(x, y, z)$ with respect to the surfaces $x+yz=0$, $y+zx=0$, and $z+xy=0$. This is an involutory birational correspondence. It is shown that if (x, y, z) are the cosines of the sides of a spherical triangle, then (x', y', z') are the cosines of the sides of a spherical triangle whose angles are the supplements of those of the given triangle. The converse is also true. The author explores the significance of this correspondence. If attention is confined to such pairs of triangles for which the ratio of the sines of the sides is fixed, the points P, P' are shown to lie on a certain fourth-order space curve. In this case results are conveniently expressed in terms of elliptic functions. A correspondence is set up between the spherical triangles and the rotations of the unit sphere. For pairs of triangles of the type mentioned, the corresponding rotations are characterized by the fact that the Euler parameters of the two rotations are reciprocals. A geometric construction for the triangles in terms of the rotation is given. *S. B. Jackson* (College Park, Md.).

***Godeaux, Lucien, et Rozet, Octave. Leçons de géométrie projective.** 2d ed. Sciences et Lettres, Liège, 1952. 278 pp. 2,800 francs.

This well-written textbook is a synthetic treatment of real projective space in the manner of Enriques [Lezioni di geometria proiettiva, 2nd ed., Zanichelli, Bologna, 1904]. Assuming the reader's familiarity with Euclidean geometry, the authors introduce ideal elements by an appeal to the analogy between points and directions. They define projective space as consisting of all the points, lines and planes, proper and ideal. After this motivation, they state nine axioms: six for incidence, one for order, one for sense, and one for continuity (Dedekind). They deduce the classical properties of duality, harmonic sets, ordered correspondences, projectivities, collineations and correlations, homologies and polarities, conics, cones, quadrics, and twisted cubics.

In the plane, they define a polarity to be an involutory correlation, and a conic to consist of the self-conjugate elements (if any) of a polarity. In space, they call an involutory correlation a null system or a polarity according as there does or does not exist a plane containing three self-polar lines. (Rather confusingly, they call polar lines "conjugate" lines.) Following Seydewitz, they define a quadric to be the locus of the point of intersection of a corresponding line and plane of two correlated bundles ("gerbes") in general position. As compared with von Staudt's treatment, this has the advantage of including cones as well as non-degenerate quadrics. It is proved later that the non-degenerate quadrics

can alternatively be derived from the self-conjugate points and planes of certain polarities, and that a general linear complex consists of the self-polar lines of a null system. Noteworthy features are the synthetic classification of collineations (pp. 80, 179–196) and the account of various ways of determining a null system (pp. 220–224).

This systematic development is interspersed with digressions of two kinds: analytic digressions, in which a knowledge of projective coordinates is assumed, and metrical digressions, in which the plane at infinity and the absolute polarity are used to supplement the reader's previous ideas of parallelism and orthogonality. No attempt is made to segregate the affine results.

Diagrams are scarce throughout most of the book, but are plentiful in the final chapter, added for this edition by the second author. Here we find a generous supply of worked exercises: projective, descriptive, affine and Euclidean. These include many problems of construction, both linear and quadratic.

H. S. M. Coxeter (Toronto, Ont.).

Terracini, Alessandro. Su certe particolarità proiettive di una coppia di coniche. Univ. e. Politecnico Torino. Rend. Sem. Mat. **10**, 259–284 (1951).

Let S and Σ be coplanar conics meeting in distinct points A, B, C, D. If pairs of tangents to Σ from points T, T' of S meet S again in the point-pairs X, Y and X', Y', then X, Y, X', Y' lie on a conic $\gamma(T, T')$ which has double contact with Σ. Let $\gamma(T, A)$, which touches S at a point on the tangent to Σ at A, meet this tangent again at A'; and let B', C', D' be similarly defined. Then A', B', C', D' are collinear for arbitrary choice of T on S if and only if either (i) S and Σ—as locus and envelope—are apolar, or (ii) S is triangularly circumscribed to Σ. The same four points are collinear (or can be so regarded) for some choice of T on S in only two other cases, (iii) where

$$\Theta'^2-4\Theta\Theta'\Delta'+12\Delta\Delta'^2=0$$

for S and Σ, with T at a residual intersection of S with a tangent to Σ at a common point of S and Σ (one of A', B', C', D' being then indeterminate and the others collinear), and (iv) when tangents to Σ at two of its intersections with S meet on S (A', B', C', D' can then coincide in pairs). Various geometrical interpretations of (iii) are obtained.

J. G. Semple (London).

Dedò, Modesto. Sui gruppi di projettività circolari. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **13(82)**, 442–450 (1949).

The linear fractional transformations of a complex variable are usually classified as elliptic, parabolic, hyperbolic and loxodromic [see, e.g., L. R. Ford, Automorphic functions, McGraw-Hill, New York, 1929, p. 23]. To avoid confusion with the classification of projectivities on the real line, the author follows Chisini in using the term "circular" instead of "elliptic". He proves that the group generated by two transformations consists entirely of circular transformations if and only if the pairs of invariant points of the two given transformations lie on a circle and separate each other. He deduces that any such group can be derived by stereographic projection from a group of rotations of a sphere. As a by-product, he obtains a new proof for the classical formula expressing one side of a spherical triangle in terms of the three angles.

H. S. M. Coxeter (Toronto, Ont.).

Süss, W. Eine selbst-duale Begründung der projektiven Geometrie von K. Menger. Elemente der Math. **7**, 29–32 (1952).

A sketch of a self-dual axiom system (i.e. a system which contains together with each axiom its dual) for the projective plane, given recently by K. Menger [Duke Math. J. **17**, 1–14 (1950); these Rev. **11**, 533], preceded by a brief historical survey of the subject. *J. L. Tits.*

Lauffer, R. Die Eulersche Gerade bei nichteuklidischer Massbestimmung. Jber. Deutsch. Math. Verein. **55**, Abt. 1, 70–76 (1952).

In the projective plane with a non-Euclidean metric, a triangle $P_1P_2P_3$ and its polar triangle $Q_1Q_2Q_3$ are in perspective from a line u whose (absolute) pole is the orthocenter H of either triangle. The trilinear poles of u for the two triangles, say S and S^*, are called their "centroids of the second kind". (In Euclidean geometry, u is the line at infinity, and S is the ordinary centroid of $P_1P_2P_3$.) The cevians P_iS meet the opposite sides in points S_i, say, and the perpendiculars to the sides at these points are found to concur in a point M called the "midpoint" of the triangle $P_1P_2P_3$. Instead of drawing perpendiculars to the sides, we may join each S_i to the point where the corresponding altitude-line P_iQ_i meets u. The three lines so obtained are found to concur in a point Z called the "center" of $P_1P_2P_3$. (In Euclidean geometry, both M and Z coincide with the circumcenter.) Similarly we can define the midpoint M^* and center Z^* of $Q_1Q_2Q_3$. The author establishes the existence of a generalized Euler line containing all the seven points: $H, S, S^*, M, M^*, Z, Z^*$. *H. S. M. Coxeter.*

Gans, David. Axioms for elliptic geometry. Canadian J. Math. **4**, 81–92 (1952).

The paper contains a stepwise reconstruction of the elliptic plane Σ by means of six axioms. Here is a sketch of it. Axiom 1: Σ is a compact metric space with at least two points. Definition and notations. xy: distance of two points x and y of Σ. Taking the diameter of Σ as unit distance, two points a and b are called "conjugate" if $ab=1$. The locus of points conjugate to any point p is termed the "polar" of p. The relation "c lies between a and b" i.e. $ac+cb=ab$, $c\neq a$, $c\neq b$ is denoted by acb or bca; c is a "midpoint" of a and b if acb and $ca=cb$. Axiom 2: Any distinct points a, b have just one midpoint if non-conjugate, just two if conjugate. Axiom 3: If abc and abd, either $c=d$, or bcd or cbd. If abc and bcd, where $ab+bc+cd\leqq 1$, then abd. Axiom 4: If c is a midpoint of a and b, then a point d exists such that $cd=1$, cad, and cbd. Here are the main consequences, all elementary: The point d described in Axiom 4 is unique; the point-set consisting of c, d and all points between c and d is called a (straight) line through a and b. Two points determine a unique line. Every line is congruent to a Euclidean circle of length 2. Σ is a S.L. (straight line) space in the sense of Busemann [Metric methods in Finsler spaces . . . , Princeton Univ. Press, Princeton, N. J., 1942; these Rev. **4**, 109], its lines and geodesics being identical. Axiom 5: Σ is two-dimensional in the sense of Menger-Urysohn. From a theorem by Busemann [loc. cit., pp. 79, 81] follows then that Σ is a projective plane. The classical fixed point theorem implies the existence for every isometry I of a point $x=I(x)$. Definitions. A translation along a line is an isometry of Σ leaving the line fixed and preserving sense on it. A rotation about a point c is an isometry leaving c fixed. If for all points x, y such that $xc=yc$ a rotation about c exists carrying

x into y, it is said that "all rotations about c exist". The strongly selective axiom is Axiom 6: Distinct lines G, H exist, each with the property that, if a, b are any points on it, there are exactly two distinct translations along it sending a into b. Consequences. All translations along G (H) have a common fixed point to be denoted by g (resp. h). All rotations about g and h exist. All rotations exist about some point of G. All rotations exist about each point of G and H. The polar of each point of G or H is a line. The group of all isometries of Σ is transitive. The polar of each point of Σ is a line. A reflection [loc. cit., pp. 113, 179] exists in each line of Σ. By a theorem of Busemann [loc. cit., p. 181], Σ is congruent to the elliptic plane. This characterization is used to prove the following statement by Busemann: If all translations exist along two geodesics of a closed S.L. plane, the metric of the latter is elliptic. *C. Y. Pauc.*

Wang, Hsien-Chung. Two-point homogeneous spaces. Ann. of Math. (2) **55**, 177–191 (1952).

Call a metric space with distance xy two-point homogeneous if for any two pairs of points x, y and x', y' with $xy=x'y'$ a motion of the space exists which carries the first pair into the second. Using the theory of transformation groups of Montgomery and Zippin extensively, the present paper establishes the following result. The only compact, convex, two-point homogeneous spaces are: the spherical spaces, the elliptic spaces, the quarternion elliptic spaces obtained by metrizing the classes of right-proportional $(n+1)$-tuples of quaternions by an expression analogous to the elliptic distance, and finally the 16-dimensional space derived in a similar manner from the right-proportional triples of Cayley numbers. Thus, for odd dimensions the spherical and elliptic spaces are the only two-point homogeneous spaces. For any dimension the elliptic space is the only one in which the geodesic through two points is unique, so that the latter property together with the other conditions characterizes the elliptic spaces. This proves an old conjecture of the reviewer [Metric methods in Finsler spaces, Princeton Univ. Press, 1942; these Rev. **4**, 109]. *H. Busemann* (Auckland).

Convex Domains, Extremal Problems, Integral Geometry

Anderson, R. D., and Klee, V. L., Jr. Convex functions and upper semi-continuous collections. Duke Math. J. **19**, 349–357 (1952).

Let f be a positive convex function defined in an open convex set D in Euclidean n-space E_n and let S_k be the set of $x \varepsilon D$ such that the hyperplanes supporting f at $(x, f(x))$ have $n-k$ degrees of freedom. The authors prove that S_k is the union of countably many sets of finite k-dimensional Hausdorff measure. The proof makes use of the collection P_f of the sets Q_x ($x \varepsilon D$), where Q_x is the set of $y \varepsilon E_n$ such that, in $E_n \times E_1$, the point nearest to $(y, 0)$ on the graph of f is $(x, f(x))$. The authors remark that P_f is an upper semi-continuous collection of compact convex sets in E_n, or as they term it, a C-collection. They show incidentally that if A_k is the subclass of a given C-collection whose members have dimension $\geqq k$, then dim $(A_k) \leqq n-k$ and moreover A_n is countable. A corollary of this last result is that no open set in E_n can be filled by a non-trivial continuous C-collection. Among the further results obtained by the authors is a generalization of a result of Borsuk (for the sphere): If a strictly convex body in E_n be covered by n closed sets, then at least one of them contains a pair of antipodal points. *L. C. Young* (Madison, Wis.).

Fenchel, W. On Th. Bang's solution of the plank problem. Mat. Tidsskr. B. **1951**, 49–51 (1951).

The tools constructed by Bang to solve the plank problem concerning coverings by strips (slabs, planks) of a convex body in a euclidean space [Mat. Tidsskr. B. **1950**, 49–53 (1950); also Proc. Amer. Math. Soc. **2**, 990–993 (1951); these Rev. **12**, 352; **13**, 769] are here used to fashion an alternate and somewhat more direct solution based on the following two lemmas. (1) Given a convex body K of width Δ and a vector $2v$ of length $\delta < \Delta$; the intersection of the two translated bodies $K - \varepsilon v$ where $\varepsilon = \pm 1$ is then a convex body of width $\geqq \Delta - \delta$. (2) Given n closed strips S_k; let $2v_k$ be a vector perpendicular to S_k whose length is the width δ_k of S_k. There are 2^n sequences $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)$ with $\varepsilon_k = \pm 1$. To each such sequence ε is associated the vector $\varepsilon v = \sum \varepsilon_k v_k$. The lemma asserts that for any $\lambda > 1$ at least one of the λ extensions of these 2^n vectors εv lies outside all n strips S_k. It is easily shown from these two lemmas that a convex body K of width Δ is not covered by n strips S_k of respective widths δ_k if $\sum \delta_k < \Delta$. For by n successive applications of lemma (1) it follows that the intersection K_n of the 2^n translated bodies $K - \varepsilon v$ is a convex body of width $\geqq \Delta - \sum \delta_k > 0$ and hence contains interior points. The origin may thus be assumed interior to K_n. All 2^n vectors εv are then interior to K, so for some $\lambda > 1$ their λ extensions are also interior to K. But according to lemma (2) at least one of these extensions must lie outside all n strips S_k. Thus the strips do not cover K. *W. Gustin* (Bloomington, Ind.).

Balk, M. B. On the division of a space of an arbitrary number of dimensions into parts by spheres (solution of an elementary problem). Uspehi Matem. Nauk (N.S.) **7**, no. 1(47), 151–154 (1952). (Russian)

The author computes the maximal number

$$N = N(n, m) \leqq 2^n$$

of domains in Euclidean m-space complementary to n spherical shells of dimensionality $m-1$, and he obtains a necessary and sufficient condition for N to be attained by given spherical shells. The author's expression for N may be interpreted as the result of adding the first $m+1$ terms of the binomial expansions of $(1+1)^n$ and $(-)^n(1-1)^n$. *L. C. Young* (Madison, Wis.).

Lekkerkerker, C. G. Packing of spheres. Math. Centrum Amsterdam. Rapport **ZW-1951-023**, 8 pp. (1951). (Dutch) Please read the corresponding note on p. 1141

Let $A(w)$ be the maximum number of solid hyperspheres of unit radius and volume ω_n which can be packed into an n-dimensional hypercube of side w, and let $A^*(w)$ be the corresponding number when the centers of the spheres are required to form a lattice. The quantities

$$\rho_n = \lim_{w \to \infty} w^{-n} A(w) \omega_n \quad \text{and} \quad \rho_n^* = \lim_{w \to \infty} w^{-n} A^*(w) \omega_n$$

are called the packing constant and the regular packing constant, respectively. It is shown here that ρ_n and ρ_n^* exist for every integer $n \geqq 1$, and that $\rho_n \leqq (n+2) 2^{-(n+1)/2}$; this appears to be the first general non-trivial upper bound for ρ_n, although R. A. Rankin [Ann. of Math. **48**, 1062–1081 (1947); these Rev. **9**, 226] obtained bounds for ρ_2 and ρ_4, while it is known that $\rho_2 = \rho_2^* = \pi/\sqrt{12}$. (Clearly $\rho_n^* \leqq \rho_n$,

but only for $n=2$ is equality known to hold.) Finally, it is shown that ρ_n^* is connected with a constant occurring in the theory of positive definite n-ary quadratic forms; this was also shown by Rankin. *W. J. LeVeque.*

Ohmann, D. Ungleichungen zwischen den Quermassintegralen beschränkter Punktmengen. I. Math. Ann. 124, 265–276 (1952).

The notation of this review will be different from that of the author [who incidentally has omitted to warn the reader of his economy of brackets in his formulae (5), (5*)]. Spaces of different dimension will occur together, distinguished by suffixes. In Euclidean n-space E_n, A_n denotes any set of points, V_n the interior of the unit sphere, S_n its boundary and $m(A_n)$ the outer (or else, inner throughout). Lebesgue measure of A_n; v_n denotes the constant $m(V_n)$ and k_n the constant k given by $1/k=nv_{n-1}$ $(n>1)$. Also, given a real function $f(\xi_n)$ of the unit vector ξ_n in S_n, $f^*(\xi_n)$ denotes its convex minorant (defined by the author by means of supporting planes, but actually identical with the usual infimum of finite sums $\sum a'f(\xi_n')$ where $\sum a'\xi_n'=\xi_n$ and each $a'>0$), and $I_n(f)$, $I_n(f^*)$ are the integrals over S_n of $f(\xi_n)$, $f^*(\xi_n)$.

Given a set A_n in E_n and a real function $g(A_{n-1})$ of sets A_{n-1} in E_{n-1}, let $G(A_n)$, $G^*(A_n)$ denote respectively the numbers $k_nI_n(f)$ and $k_nI_n(f^*)$, where f is the function of unit vectors ξ_n in S_n defined by choosing for $f(\xi_n)$ the value $g(A_{n-1})$ for which A_{n-1} is the orthogonal projection of A_n onto the Euclidean $(n-1)$-space normal to the unit vector ξ_n. By induction, starting with the functions

$$g_0(A_n)=h_0(A_n)=m(A_n) \quad (n=1, 2, 3, \cdots),$$

define set functions

$$g_r(A_n), \quad h_r(A_n) \quad (r=0, 1, \cdots, n-1; n=1, 2, 3, \cdots)$$

by writing for each n, $g_r(A_n)=G(A_n)$ when $g(A_{n-1})$ is $g_{r-1}(A_{n-1})$, and $h_r(A_n)=G(A_n)$ when $g(A_{n-1})$ is $h_{r-1}(A_{n-1})$. For a fixed A_n, let now

$$w_p=g_p(A_n), \quad w^*{}_p=h_p(A_n) \quad (p=0, 1, \cdots, n-1)$$

and $w_n=w_n{}^*=v_n$.

The author states that he intends to devote a series of papers to inequalities of the type $(w_p)^{n-r}\geqq(v_n)^{p-r}(w_r)^{n-p}$ $(0\leqq r<p\leqq n)$, i.e. of a type well known in the theory of convex figures. In this first paper he deals with the case $r=0$ and establishes the stronger result $(w^*{}_p)^n\geqq(v_n)^p\,m(A_n)^{n-p}$. He remarks that for $p=1$ there follows for the $(n-1)$-dimensional Minkowski boundary-measure $\sigma(A_n)$ (which is easily seen to satisfy $\sigma(A_n)\geqq nw_1$) a strengthened isoperimetric inequality. The reviewer observes that here $\sigma(A_n)$ can be replaced without much trouble by a more satisfactory boundary-measure [$\sigma(A_n)$ is apt to be rather large except for very simple sets A_n]. *L. C. Young.*

Wintner, Aurel. On isometric surfaces. Amer. J. Math. 74, 198–214 (1952).

A vectorial representation $X(u, v)$ of a surface S in E^3 is of class C^n (analytic) if $X(u, v)$ possesses continuous partials of order n and the vectors $\partial X/\partial u$ and $\partial X/\partial v$ are everywhere independent. Let S': $X'(u, v)$ be a second surface of class C^n, $n\geqq 1$, (or an analytic surface). If for sufficiently small parameter regions a transformation $u'=u'(u, v)$, $v'=v'(u, v)$ of class C^1 with non-vanishing Jacobian exists such that $|dX'(u', v')|\equiv|dX(u, v)|$, then S and S' are isometric and the transformation is an isometry. We say the isometry is of class C^n (analytic) if $u'(u, v)$ and $v'(u, v)$ are of class C^n (analytic). The paper deals with the question: what can be said about the class of the isometries, when S and S' are of a certain class. For instance, the analyticity properties of conformal mappings yield: if S and S' are analytic, then all their isometries are analytic. Under certain Hölder conditions for the nth derivatives, results of Lichtenstein permit one to extend this to surfaces of class C^n. Examples are given to make it plausible that this is probably not true without the Hölder conditions and also examples which show that the situation is more complicated if the surfaces S, S' are given abstractly, instead of imbedded in E^3.

The longest discussion concerns the following theorem: Two closed essentially convex (=convex with Gauss curvature positive except at a set of measure 0) surfaces in E^3 of class C^2 are congruent when they are C^2-isometric. This had been proved very elegantly by Herglotz in 1943 [Abh. Math. Sem. Hansischen Univ. 15, 127–129 (1943); these Rev. 7, 322]. Since that time the theorem has been freed from all differentiability hypotheses by the Russian school. An older result of Pogorelov showed rather simply that it suffices to assume that one of the surfaces has the property that the ratio of the area of a set on the unit sphere to the arc of its image on the surface under the spherical mapping is bounded. A more recent result, by the same author [Doklady Akad. Nauk SSSR (N.S.) 79, 739–742 (1951); these Rev. 13, 271], for which a complete proof has not yet been published, deals with complete arbitrary closed convex surfaces. Will the time ever come, when the existence of these papers is admitted elsewhere? *H. Busemann.*

Ueno, Seitaro, Hombu, Hitoshi, and Naito, Jun. Some integral-geometric inequalities. Mem. Fac. Sci. Kyūsyū Univ. A. 6, 97–106 (1951).

If Γ is the sum of finite or countably infinite convex domains with disjoint inner points in a unit circle such that its intersection with any straight line is of length $<\delta$, the area of Γ does not exceed 2δ. Feller proved that this is in a sense the best possible estimate [Duke Math. J. 9, 885–892 (1942); these Rev. 4, 168]. This result is extended to the non-Euclidean plane and then to n-dimensional non-Euclidean spaces. For instance, on the sphere the authors proved that the area is $<2\delta\sin\alpha$, α being the radius of the geodesic circle in which Γ lies, and that this is the best possible estimate. Similar results hold for n dimensions.
S. Chern (Chicago, Ill.).

Chern, Shiing-shen. On the kinematic formula in the Euclidean space of n dimensions. Amer. J. Math. 74, 227–236 (1952).

In vorliegender Arbeit wird die kinematische Hauptformel der Integralgeometrie für einen n-dimensionalen euklidischen Raum hergeleitet. Gegeben seien zwei orientierbare und zweimal differenzierbare Hyperflächen Σ_0 und Σ_1 des euklidischen Raumes. Mit $V_i(i=0, 1)$ werden die Rauminhalte der von $\Sigma_i(i=0, 1)$ begrenzten Gebiete D_i bezeichnet. Es werde nun Σ_0 festgehalten und Σ_1 so bewegt, dass es mit Σ_0 einen gemeinsamen Durchschnitt besitzt. Von den Hyperflächen wird dabei vorausgesetzt, dass der Durchschnitt D_0, D_1 höchstens aus einer endlichen Anzahl von Teilbereichen besteht. Unter diesen Voraussetzungen existiert die Euler-Poincaré'sche Charakteristik $\chi(D_0D_1)$ des Durchschnittes D_0D_1. Ist I_{n-1} die Oberfläche der Einheitshypersphäre, dann betrachtet Verf. neben $\chi(D_0D_1)$ noch die Eckenkrümmung des Durchschnittes von D_0D_1 die durch $K(D_0D_1)=I_{n-1}\chi(D_0D_1)$ definiert ist. Ist Σ_1 die kinematische

Dichte von Σ_1, dann ist das Integral der Hauptformel durch das über die oben beschriebenen Lagen von Σ_1 erklärte Integral $\int K(D_0D_1)\dot{\Sigma}_1$ bestimmt. Dieses Integral kann nun durch eine Reihe von Invarianten der Σ_i ausgedrückt werden, die wir jetzt angeben. Es bezeichne dA das Oberflächenelement und S_i die elementalsymmetrischen Funktionen der Hauptkrümmung eines Punktes von Σ. Setzt man $S_0=1$, dann seien $M_i=\binom{n-1}{i}\int_\Sigma S_i dA$ $(i=0,\cdots,n-1)$. Bei Verwendung beider Hyperflächen werden diese Invarianten mit $M_i^{(0)}$ und $M_i^{(1)}$ bezeichnet. Führt man noch $J_n=I_1I_2\cdots I_{n-1}$ ein, dann erhält man die kinematische Hauptformel in der Gestalt

$$\int K(D_0D_1)\dot{\Sigma}_1 = J_n\left\{M_{n-1}^{(0)}V_1+M_{n-1}^{(1)}V_0+\frac{1}{n}\sum_{k=0}^{n-2}\binom{n}{k+1}M_k^{(0)}M_{n-2-k}^{(1)}\right\}$$

Der Beweis stützt sich auf die Anwendung der Cartanschen Methode des beweglichen Bezugssystems. Ohne auf Einzelheiten des Beweises einzugehen erwähnen wir eine schon wegen seines Zusammenhanges zur Theorie der gefaserten Räume bemerkenswerte Begriffsbildung. Sie besteht in der Einführung einer Dichte für eine solche Schar von k-Beinen eines orientierbaren Riemannschen Raumes eine Klasse die nicht kleiner als drei ist, die aus einem Punkt und k von demselben ausgehenden paarweise orthogonalen Einheitsvektoren besteht. *O. Varga* (Debrecen).

Santaló, Luis Antonio. On some integral formulas and mean values concerning movable convex figures in the plane. Univ. Buenos Aires. Contrib. Ci. Ser. A. **1**, 23–45 (1950). (Spanish)

Let K_0 be a fixed bounded closed convex set with interior points in the plane and let $K_1,\cdots,K_n$ be variable sets of the same type which are congruent to each other. The author generalizes some of his earlier results [Abh. Math. Sem. Univ. Hamburg **11**, 222–236 (1936)] as follows: Denote by $A_0/2\pi$ and $A/2\pi$ the areas of K_0 and K_i, by U_0 and U their perimeters and put $T=A_0+UU_0$. Let σ_r be the subset of K_0 which for a definite position of $K_1,\cdots,K_n$ lies in r of these sets, $1\leqq r\leqq n$. If f_r is the arc, u_r the total perimeter and N_r the number of components of σ_r, then

$$\int f_r dK_1\cdots dK_n=\binom{n}{r}A^rT^{n-r}A_0/2\pi,$$

$$\int u_r dK_1\cdots dK_n=\binom{n}{r}A^rT^{n-r}U_0+n\binom{n-1}{r}A^{n-1}T^{n-r}A_0U + n\binom{n-1}{r}A^rT^{n-r-1}A_0U,$$

$$\int N_r dK_1\cdots dK_n=nA^rT^{n-r-1}A_0\left[\binom{n-1}{r}A+\binom{n-1}{r-1}T\right] +\binom{n}{r}A^rT^{n-r}+\binom{n}{2}u^2A_0A^{r-2}T^{n-r-2} \times\left[\binom{n-2}{r-2}T^2+\binom{n-2}{r-1}2AT+\binom{n-2}{r}A^2\right] +nUU_0A^{r-1}T^{n-r-1}\left[\binom{n-1}{r-1}T+\binom{n-1}{r}A\right],$$

where the integrals are extended over all positions of $K_1,\cdots,K_n$ which intersect K_0. These results are applied to obtain expressions for the same integrals when $K_1,\cdots,K_n$ are strips between parallel lines of the same width. *H. Busemann* (Auckland).

Algebraic Geometry

Tibiletti, Cesarina. Costruzione delle curve multiple risolubili prive di punti di diramazione: caso generale. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **14(83)**, 619–636 (1950).

Sur la riemannienne R_f d'une courbe $f(x,y)=0$, la construction d'une fonction $z(x,y)$ dont le groupe G est résoluble et dont on assigne les substitutions sur les cycles de R_f, peut se ramener à la construction de courbes multiples cycliques donnée par Chisini [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) **24**, 153–158 (1915)]. La construction se fait à partir de celles des fonctions que l'auteur nomme normales, dont l'ordre égale le nombre des opérations du groupe; pour celles-ci on se ramène à des extractions successives de radicaux sur des riemanniennes successives. Dans le cas général, pour construire une fonction à n valeurs, correspondant à un groupe G transitif résoluble d'ordre kn, on construit la fonction normale à kn valeurs, w; alors la fonction cherchée s'obtient à l'aide d'une fonction symétrique de k variables auxquelles on substitue les k éléments de w, correspondants à une colonne de la décomposition de G à partir du sous-groupe d'ordre n qui laisse fixe l'élément unité. *B. d'Orgeval* (Alger).

Godeaux, Lucien. Sur quelques involutions rationnelles appartenant à une surface algébrique. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **38**, 244–252 (1952).

Construction d'une involution cyclique, appartenant à une surface de genres $p_a=p_g=4$, possédant quatre points unis, dont l'image est une surface rationnelle.

Author's summary.

d'Orgeval, Bernard. À propos d'une surface du quatrième ordre. Bull. Soc. Roy. Sci. Liège **20**, 437–438 (1951).

In a paper of C. F. Manara [Boll. Un. Mat. Ital. (3) **6**, 9–14 (1951); these Rev. **12**, 735] a surface F_4 of order four with a double line and two conical and two biplanar double points was constructed. The present note offers two other proofs of the existence of such a surface. It also includes an incomplete proof of the irreducibility of the branch curve of the triple plane obtained by projecting F_4 from one of its simple points. [See the following review.]

H. T. Muhly (Iowa City, Iowa).

Manara, Carlo Felice. Sur une démonstration d'irréductibilité. Bull. Soc. Roy. Sci. Liège **20**, 675–676 (1951).

The author calls attention to the inconclusive argument mentioned in the preceding review and offers an alternative short proof. *H. T. Muhly* (Iowa City, Iowa).

Spampinato, Nicolò. Rappresentazioni complesse delle trasformazioni birazionali di un S_r ipercomplesso. Ricerca, Napoli **3**, no. 1, 8–12 (1952).

Dantoni, Giovanni. Due dimostrazioni elementari dell'esistenza di modelli birazionali privi di punti multipli di una curva algebrica, con applicazioni alle superficie. Ann. Scuola Norm. Super. Pisa (3) **5**, 355–365 (1951).

For an irreducible algebraic h-dimensional variety V let $S(V)$ be the set of those birational transforms of V, of

order ν and spanning a space of dimension ρ, for which (*) $\nu \leqq 2(\rho-h)$. Theorem A: The projection of any member of $S(V)$ from any of its singular points is a member of $S(V)$. It follows that if V_0 is a member of $S(V)$ with minimum ρ, then V_0 is non-singular. Theorem B: If $h=1$, $S(V)$ is not empty. Corollary: Every irreducible curve has a non-singular birational transform. The proof of Theorem B does not carry over for $h>1$, but does for $h=2$ if (*) is replaced by $\nu \leqq 3(\rho-2)$. Now one can prove Theorem A only for varieties which carry no involution of order two. The Corollary therefore holds for surfaces of this type.

The second proof considers the set of all birational transforms of an irreducible curve C which span spaces of a given dimension ρ and the minimum order $\bar{\nu}(\rho)$ of the curves of this set. The integer-valued function $M(\rho)=\bar{\nu}(\rho)-\rho$ is shown to be non-decreasing and bounded; hence there exist integers $\bar{p}$ and $\bar{\rho}$ such that $M(\rho)<\bar{p}$ if $\rho<\bar{\rho}$ and $M(\rho)=\bar{p}$ if $\rho \geqq \bar{\rho}$. Theorem C: A birational transform of C spanning a space of dimension $\rho \geqq \max(\bar{\rho}, \bar{p})$ and of order $\bar{\nu}(\rho)$ is non-singular. Since there exist birational transforms of C spanning spaces of arbitrarily high dimension, the Corollary follows. $\bar{p}$ turns out to be the genus of the curve. *R. J. Walker.*

Turri, Tullio. Sopra una recensione di nota relativa a tabelle dei periodi di integrali abeliani reali. Rend. Sem. Fac. Sci. Univ. Cagliari **20** (1950), 171–174 (1951).

L'A. fa qualche breve commento chiarificatore ad una recensione di un suo lavoro [stessi Rend. **17**, 29–33 (1948); questi Rev. **11**, 685] fatta dallo scrivente, senza invalidarne la sostanza. *F. Conforto* (Roma).

Derwidué, L. Rectification. Math. Ann. **124**, 316 (1952).

As a result of a counter-example sent to the author by B. Segre, he wishes to state that the main results of the following papers rest upon an unproved hypothesis: Math. Ann. **123**, 302–330 (1951); **124**, 65–76 (1951) [these Rev. **13**, 67, 488]; and the paper reviewed below.

Derwidué, L. Sur les points unis des involutions cycliques. Mém. Soc. Roy. Sci. Liège (4) **11**, 5–52 (1951).

The following main theorem is stated: Let T be a cyclic birational transformation of an irreducible k-dimensional algebraic variety V into itself. V has a non-singular birational transform on which the fixed points of the image of T form a finite set of disjoint non-singular $(k-1)$-dimensional varieties. The proof makes essential use of the results and methods of a previous paper of the author's [Math. Ann. **123**, 302–330 (1951); these Rev. **13**, 67], some of which have been shown to be incorrect [see the cited review]. *R. J. Walker* (Ithaca, N. Y.).

Derwidué, L. Sur une transformation crémonienne. Mém. Soc. Roy. Sci. Liège (4) **11**, no. 3, 33 pp. (1951).

A simplification and extension of some results of an earlier paper of the author's [same Mém. (4) **9**, no. 2 (1949); these Rev. **11**, 740]. The criticisms referred to in the preceding review apply also to this earlier paper. *R. J. Walker.*

Benedicty, Mario. La base sulla varietà di Segre e su varietà analoghe. Rend. Soc. Ital. Sci. Accad. dei XL (3) **27**, 233–243 (1949).

The author determines, by simple geometrical means, the base on (i) complex Segre varieties, (ii) real Segre varieties, (iii) the symmetric n-fold product of an S_r with itself; and shows that these varieties are of minimal order with respect to topological (and not merely birational) transformations. *J. A. Todd* (Cambridge, England).

Brusotti, Luigi. Questioni di realità e modelli algebrici. Univ. e Politecnico Torino. Rend. Sem. Mat. **10**, 139–153 (1951).

A summary of results obtained by the author and others on real varieties is given. The problem of representing a finite system of looped circuits as the real part of a twisted real curve receives particular attention. *H. T. Muhly.*

Andreotti, Aldo. Sopra le varietà di Picard d'una superficie algebrica e sulla classificazione delle superficie irregolari. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **10**, 380–385 (1951).

Results on the problem of classification of irregular surfaces are presented here, with only sketches of proofs. Let F be a surface of irregularity q; then two Abelian varieties, V_q and V_q', are attached to F: V_q has the same matrix Ω of periods as the simple integrals of the 1st kind of the surface F; V_q' represents the linear systems of an ∞^q continuous system on F. The A. determines the matrix Ω' of V_q' as a function of Ω; hence it follows that V_q' is birationally equivalent to V_q when all the elementary divisors of Ω are 1, while in every other case Ω and Ω' are (always isomorphic but), in general, not equivalent, so that V_q and V_q' are birationally distinct. Moreover, the A. generalizes a theorem by Severi (who considered all the simple integrals of the 1st kind on F) to the case when only a regular system of irreducible integrals is taken into consideration. From this theorem it follows that, on V_q, there is a rational image of F, Φ; if F does not contain any pencil of genus q, Φ is also a surface; and in this case, the A. conjectures that Φ is birationally equivalent to F, provided F is not an elliptic surface with torsion. The A. has, as yet, only the proof for $q=3$, and when all the elementary divisors of Ω are equal to 1. Consequently, the classification of the irregular surfaces with $q=3$, which do not contain any pencil of genus 3 and are not elliptic (with torsion), in case all the elementary divisors of Ω are 1, is reduced to classify the surfaces in an Abelian V_3; and this problem may be solved using the theory of theta-functions, also for any elementary divisors (not only in case they are all 1). In fact, the A. gives the numeric values of the genera and plurigenera for the families of surfaces obtained by setting equal to zero the theta-functions of the different systems; and for each family the number of moduli is also obtained. If $q \geqq 4$, a surface in an Abelian V_4 can obviously no longer be obtained by setting equal to zero a single theta-function; the A. considers therefore theta-ideals for which the "Maximalbedingung" holds and the technique of syzygies may be used. Every surface on V_4 may be defined by a theta-ideal; one is then able to compute the Hilbert function, whence the values of invariants follow. But new difficulty occurs when one tries to find the complete systems of these ideals. For $q=4$ complete results are still presented for surfaces of a V_4 that are either complete intersections or of finite residual. In general, one may only say that for $q \geqq 4$ the problem of classifying the irregular surfaces is very similar to the problem of classifying the curves in a linear space; even Halphen's conjecture of the necessity of an infinite number of invariants has here its analog. From these researches it appears, moreover, that Abelian varieties, rather than projective spaces, are the natural medium for studying the irregular surfaces. *F. Conforto* (Rome).

Weil, André. Criteria for linear equivalence. Proc. Nat. Acad. Sci. U. S. A. **38**, 258–260 (1952).

Extension to divisors on an arbitrary normal variety V of Severi's results about curves on a nonsingular surface. Two kinds of criteria for linear equivalence (denoted $\sim$) are given. The criteria of the first kind state that a divisor X rational; over k is ~ 0 on V (or that $mX \sim 0$, m being an integer $\neq 0$) provided its trace on a suitable subvariety W of V depending on X is ~ 0 on W (for W one may take the section of V by a linear variety generic over k). In the criteria of the second kind, X is already supposed to be algebraically equivalent to 0 on V, and is proved to be ~ 0 (or $mX \sim 0$) provided its trace on a fixed subvariety W is ~ 0 on W. When W is a curve C, this last condition is linearly equivalent to: for every rational mapping f of V into an abelian variety, the sum $S(f(X \cdot C))$ is 0. These results open the way to a purely algebraic theory of the Picard varieties. Their proofs will be published in a forthcoming paper.

P. Samuel (Ithaca, N. Y.).

Northcott, D. G. Specializations over a local domain. Proc. London Math. Soc. (3) **1**, 129–137 (1951).

Let R be a local domain (i.e., a local ring without zero divisors), M its maximal ideal, $K = R/M$, F the quotient field of R. Let $(y) = (y_1, \cdots, y_n)$ and $(\alpha) = (\alpha_1, \cdots, \alpha_n)$ be n-tuples over extension fields of F and K respectively. Then (α) is called a specialization of (y) over R if the natural homomorphism $R \to K$ can be extended to a homomorphism $R[y] \to K[\alpha]$ with $y_i \to \alpha_i$. One of the main results is as follows: Let R be analytically irreducible (i.e., its completion has no zero divisors), let $y_i \in F$, let (α) be a specialization of (y) over R such that (α) is algebraic over K, and assume that for every specialization (β) of (y) over R such that (α) is a specialization of (β) over K, we have (β) is algebraic over K. Then any specialization of (y) over R is a conjugate of (α) over K. The proof requires some theorems on specializations over complete local domains. These theorems have been proved by Weil [Foundations of algebraic geometry, Amer. Math. Soc. Colloq. Publ., vol. 29, New York, 1946; these Rev. **9**, 303] in the special case of power series rings; see theorems 1 and 3 of chapter 3.

If V and V' are birationally equivalent algebraic varieties, let P be a point on V, R the quotient ring of P, (y) the general point of V'. Then a point (α) of V' corresponds to P if and only if (α) is a specialization of (y) over R. This observation, together with the result stated above, yields a new proof of Zariski's "main theorem" on birational correspondences [Trans. Amer. Math. Soc. **53**, 490–542 (1943); these Rev. **5**, 11]. *I. S. Cohen* (Cambridge, Mass.).

Gröbner, Wolfgang. L'ideale aggiunto di una varietà algebrica. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **10**, 57–63 (1951).

An assertion of the author [Hamburger Math. Einzelschr. **30** (1941); these Rev. **5**, 155] to the effect that the conductor of the ring of homogeneous coordinates of a variety V_d/k (k algebraically closed and of characteristic zero) with respect to its integral closure can be used to define adjoints (in case $d=1$) and subadjoints (in case $d>1$) of V_d is here elaborated. The contention is supported in case $d=1$, and the possibility that the case $d>1$ can be treated analogously is suggested.

In view of the theorems used, it seems strange to the reviewer that no reference is made to the work of Zariski since 1940, nor for that matter to the work of Dedekind and Weber [J. Reine Angew. Math. **92**, 181–290 (1882)] where among other things, the connection between conductors and adjoints is discussed. *H. T. Muhly* (Iowa City, Iowa).

Differential Geometry

van der Waag, E. J. Sur les plans osculateurs. I. Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 41–51 (1952).

L'auteur poursuit son analyse fine des notions fondamentales de la géométrie différentielle [C. R. Akad. Sci. Paris **231**, 1026–1027, 1120–1122 (1950); Nederl. Akad. Wetensch. Proc. Ser. A. **54**, 390–417 (1951); ces Rev. **12**, 633, **13**, 771]. Notations—k: courbe continue ("path" selon Menger); O: point fixe sur k, origine du système de coordonnées (x, y, z); P, P', P'', P''': points variables sur k; t_P ($t_{P'}$): tangente à k en P (resp. P') si elle existe; t_P': droite passant par O, parallèle à t_P; $t'_{P''}$: droite passant par P' et parallèle à $t_{P''}$. Huit définitions de plan osculateur π de k en O sont données:

(I) $\pi = \lim \text{plan}(t_O, P)$ pour $P \to O$;
(II) $\pi = \lim \text{plan}(P', O, P'')$ pour P' et P'' tendant vers O de part et d'autre de O;
(III) $\pi = \lim \text{plan}(t_O, t_P')$ pour $P \to O$;
(IV) $\pi = \lim \text{plan}(O, P', P'')$ pour P' et $P'' \to O$;
(V) $\pi = \lim \text{plan}(O, t_P)$ pour $P \to O$;
(VI) $\pi = \lim \text{plan}(P', P'', P''')$ pour $P', P'', P''' \to O$;
(VII) $\pi = \lim \text{plan}(P', t_{P''})$ pour P' et $P'' \to O$;
(VIII) $\pi = \lim \text{plan}(t_{P'}, t'_{P''})$ pour P' et $P'' \to O$.

Dans chaque cas ne sont acceptées que les configurations (t_O, P), (P', O, P''), $\cdots$ définissant un plan unique, ce qui implique les restrictions suivantes: (I),(III),(V) $P \neq O$, (IV) $P' \neq O$, $P'' \neq O$, $P' \neq P''$, (VI) $P' \neq P''$, $P'' \neq P'''$, $P''' \neq P'$, (VII),(VIII) $P' \neq P''$. S'il existe un voisinage de O sur k tel que, sous une de ces conditions, le plan générateur y soit partout défini, le plan osculateur correspondant est dit "défini avant la limite". [Remarque: Pour rendre les définitions précédentes précises si k n'est pas un arc simple, il est nécessaire de distinguer entre un "point sur k" [Pauc, Actualités Sci. Ind., no. 885, Hermann, Paris, 1941; ces Rev. **7**, 67] et son support.] De l'abondant matériel, le rapporteur a extrait les résultats qui lui ont semblé les plus caractéristiques, en particulier d'instructifs contre-exemples. Si k possède une tangente ordinaire dans un voisinage de O et si (III)π existe et est défini avant la limite, (I)π existe. La condition que (III)π soit défini avant la limite est indispensable comme le montre la courbe $z = x^3 \sin x^{-1} = \varphi(x)$, $y = \int_0^x \varphi'^2(\xi)\, d\xi$. La non-implication de V par III est montrée sur l'exemple $z = x\zeta(x)$, $y = \int_0^x \xi z'(\xi)\, d\xi$ où

$$\zeta'(x) = x(1 + x - \sin x^{-1})).$$

Pour la courbe $x = t^3(2 - \sin t^{-1})$, $y = t^4$, $z = t^4(2 - \sin t^{-1})$, (IV)π est défini avant la limite, (III)π n'existe pas. Un tel phénomène n'est pas possible si $y = f(x)$, $z = g(x)$.

C. Pauc (le Cap).

van der Waag, E. J. Sur les plans osculateurs. II. Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 52–62 (1952).

(Suite de l'analyse précédente.) Si k possède partout dans un voisinage de O une tangente ordinaire et si (V)π existe et est défini avant la limite, les plans (IV) à droite et à

gauche existent, définis aussi avant la limite, (I)π existe. [Remarque. Il semble au rapporteur que (IV)π existe et est = (V)π. Dans l'exemple $y=x^4$, $z=x^3$, la tangente en O est de rebroussement. Plus loin en 4.7.2., sous les mêmes hypothèses sur k, l'auteur affirme que les demi-tangentes en O existent. Or d'après la définition donnée en 3.0., l'existence d'une tangent ordinaire en O équivaut à l'existence des deux demi-tangentes (antérieure et postérieure) opposées en O.] Si k possède partout dans un voisinage de O une tangente ordinaire et (V)π existe, (I)π peut ne pas exister, même si k admet une représentation $y=f(x)$, $z=g(x)$. Lorsque le plan à droite VI existe, étant défini avant la limite, la demi-tangente à droite existe et, en prenant celle-ci comme axe des x, la courbe peut être représentée à droite de O par $y=f(x)$, $z=g(x)$. Si partout dans un voisinage du point O, la courbe possède une tangente ordinaire et si (VII)π existe, étant défini avant la limite, la courbe k admet une représentation $y=f(x)$, $z=g(x)$ dans un voisinage de O, f et g admettant des dérivées f' et g' continues pour $x=O$. Lorsque partout dans un voisinage de O, k possède une tangente ordinaire, (VIII)π étant défini avant la limite, (VII)π existe aussi et est défini avant la limite. Si k admet une représentation $p=\mathbf{p}(t)$ possédant pour $t=0$ une dérivée seconde au sens large $\mathbf{p}''(0)$ telle que le produit vectoriel $\mathbf{p}'(0)\times\mathbf{p}''(0)\neq\mathbf{0}$, tous les plans osculateurs (I)π, $\cdots$, (VIII)π existent et sont définis avant la limite au point $O=\mathbf{p}(0)$.

C. Pauc (le Cap).

Mirguet, Jean. Sur la convexité d'un domaine, extérieur à la véritable double courbure. C. R. Acad. Sci. Paris 234, 398–399 (1952).

Cette note poursuit des travaux antérieurs [Revue Sci. 85, 67–72 (1947); C. R. Acad. Sci. Paris 232, 1632–1634 (1951); ces Rev. 9, 18; 13, 70]. S désigne une orthosurface à paratingent supérieur fini, (T) l'ensemble des points de S où le biptg. contient toutes les directions de plans de l'espace, (A) l'extérieur sur S de l'ensemble des points où S possède la véritable double courbure. L'auteur montre que tout point M de (A) dans le voisinage duquel (T) est dénombrable, est intérieur à un domain convexe de (A). Une autre condition locale est donnée permettant la même conclusion.

C. Pauc (le Cap).

Bouligand, Georges. Sur les transformations de contact de l'espace. C. R. Acad. Sci. Paris 234, 908–910 (1952).

$M=\mathbf{M}(x, y, z, p, q)=M(m, \varpi)$ représente une transformation de contact dans l'espace euclidien à trois dimensions, définie dans un voisinage de l'élément du premier ordre (m_0, ϖ_0), telle que les vecteurs $\mathbf{M}_x+p\mathbf{M}_z$, $\mathbf{M}_y+q\mathbf{M}_z$, $\mathbf{M}_p$ et $\mathbf{M}_q$ soient fonctions continues de (m, ϖ) et définissent un plan $P=\mathbf{P}(m, \varpi)$. L'auteur définit une notion de régularité pour les suites de triplets $(m, \varpi)=(m_0, \varpi_0)$, (m', ϖ'), (m'', ϖ'') convergeant vers (m_0, ϖ_0) telle que le plan défini par les images $M=M_0$, M' et M'' tende vers $P_0=\mathbf{P}(m_0, \varpi_0)$.

C. Pauc (le Cap).

Haupt, Otto, Über eine Kennzeichnung von Bogen minimalen Ordnungswertes. Gaz. Mat., Lisboa 12, no. 50, 23–26 (1951).

Let k be a fixed integer for which $k\geqq 2$. An order characteristic is a member of a set $\mathfrak{k}$ of curves defined in the Euclidean plane such that exactly one member of $\mathfrak{k}$ goes through k given points of the plane and that curve varies continuously as the k points vary continuously. Let B be an open arc for which an interior and an exterior is defined by completing it to a simple closed curve. If the order characteristic through k points of B converges to an order characteristic as these k points all converge to a single point q, this limit position is defined to be a paratingente with point of contact q. A curve C of $\mathfrak{k}$ is defined to be normally ordered with respect to B if a direction can be assigned to C so that a point moving continuously along C in this direction encounters the intersection points of B and C in the same order as they are encountered when a point moves continuously along B in a fixed direction. The concept of normal ordering is extended to a paratingente by continuity considerations. If a paratingente C with point of contact q is normally ordered, every curve neighborhood of C consists of q, a forward arc and a rear arc. Two normally ordered paratingentes with points of contact q, q' are defined to be of similar type at q and q' if the forward (rear) arcs of two sufficiently small curve neighborhoods of q, q' both (1) contain only points interior to B, or (2) contain only points exterior to B, or (3) intersect B infinitely many times and contain points interior to B as well as points exterior to B, or (4) have infinitely many points in common with B but contain no points exterior to B, or (5) have infinitely many points in common with B but contain no points interior to B.

The author announces the following theorem: B is an open arc no subarc of which is an arc of a member of $\mathfrak{k}$ and such that each member of $\mathfrak{k}$ is normally ordered with respect to B, then (1) implies (2) and (2) implies (1), where (i) no member of $\mathfrak{k}$ cuts B in more than k points, (ii) all paratingentes are of similar type at their points of contact. No proof is given. Two examples are discussed. In the first $\mathfrak{k}$ consists of all straight lines and $k=2$; in the second $\mathfrak{k}$ consists of all circles including lines as special cases and $k=3$.

D. Derry (Vancouver, B. C.).

Cattaneo, Carlo. Sul legame lineare che intercorre fra le tre forme quadratiche associate a una superficie. Boll. Un. Mat. Ital. (3) 7, 16–17 (1952).

There are three quadratic forms associated with a surface: $\alpha=ds^2$; $\beta=b_{hk}dx^hdx^k$; $\gamma=c_{hk}dx^hdx^k$. These are related by the identity: (1) $\gamma=-K\alpha-H\beta$. Further, the principal radii of curvature of the surface satisfy the equation: (2) $-1/\rho^2+H/\rho-K=0$. Consider a curve C cut from the surface by a plane through a normal at P. Let $d\chi$ be the angle between the principal normal to C at a nearby point P' and the normal to the surface at P'. Then the author proves: (3) $\gamma=\alpha^2/\rho^2+d\chi^2$ where ρ is the radius of curvature of C at P. From (1) and (3) it follows that

$$(4)\qquad (d\chi/ds)^2=-1/\rho^2+H/\rho-K$$

which gives an interpretation of the left side of (2).

C. B. Allendoerfer (Seattle, Wash.).

Mishra, R. S. On the congruence of curves through points of a subspace imbedded in a Riemannian space. Ann. Soc. Sci. Bruxelles. Sér. I. 65, 109–115 (1951).

In an as yet unpublished paper, the writer has generalized the definition of union curves to apply to curves in a subspace of a Riemann space and obtained their differential equation. Using this result, the present paper proceeds to obtain a tensor expression which is defined to be the union curvature vector. This vector is zero if and only if the corresponding curve is a union curve.

A. Fialkow.

Dubnov, Ya. S. Straight-line congruence of an affine gradient. Doklady Akad. Nauk SSSR (N.S.) 81, 349–352 (1951). (Russian)

Given a scalar field $\varphi(M)$, where M is a point with radius vector r on a surface in three-dimensional affine space. Then the end-points of the vectors $dr/d\varphi$ at a point M lie on a straight line parallel to the tangent line to $\varphi=\text{const}$. This line is called the affine gradient of the field φ at point M. When we pass to the metrical case and MP is the metrical gradient of φ, then P is the pole of the affine gradient with respect to the unit circle around M in the tangent plane. It is shown that if to every point of the surface is associated a line in the tangent plane, then the necessary and sufficient condition that the congruence of lines thus formed admits a gradient is that its developable surfaces correspond to a conjugate set on the surface. Several examples are discussed in connection with functions affinely related to the surfaces, such as the so-called Čebyšev tensors. In the case of the plane the correspondence between points and lines is such that when a point M_0 moves on a line MK and the corresponding line $l(M_0)$ describes a class curve tangent to l at L, then the projectivity $M_0K\to ML$ is involutory. In the particular case of the polar involution with respect to a conic, of center O, $\varphi(M)=-\frac{1}{2}\ln|(OM/ON)^2-1|$, where N is the point where the ray OM intersects the conic.

D. J. Struik (Cambridge, Mass.).

Hartman, Philip. On unsmooth two-dimensional Riemannian metrics. Amer. J. Math. 74, 215–226 (1952).

It is shown that the metric $du^2+2\cos\phi\,du\,dv+dv^2$, having "curvature" -1 and continuous ϕ may be imbedded in a Euclidean 3-space and that this imbedding is unique, up to Euclidean motions, if the parametric curves are to be asymptotic lines. It is also proved that any transformation $u^i=u^i(U^j)$ $(i,j=1,2)$ of class C^1 which transforms a positive definite form $g_{ij}du^idu^j$ into another form $G_{ij}dU^idU^j$ with the same properties is necessarily of class C^2.

A. Fialkow (Brooklyn, N. Y.).

Castoldi, Luigi. V_m "canoniche" in V_n e loro caratterizzazione intrinseca. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 13(82), 267–279 (1949).

The author calls a V_m imbedded in V_n canonical of order r if there exist at most r vectorfields v^h, linearly independent and normal to V_m, such that the covariant differential δv^h is tangent to V_m. This order can be determined algebraically in terms of the number of linearly independent solutions of a set of linear homogeneous equations (the integrability conditions).

A. Nijenhuis (Amsterdam).

Pratelli, Aldo M. Sui tensori emisimmetrici coniugati. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 13(82), 473–487 (1949).

Given an alternating tensor $V_{i_1i_2\cdots i_p}$ in a Riemannian n-space (skew-symmetric for an interchange of any two of its p suffixes), its conjugate is defined by

$$*V^{i_1\cdots i_r}=\frac{1}{(n-r)!}\epsilon^{h_1\cdots h_n}V_{h_1\cdots h_{n-r}},$$

where $r=n-p$ and ϵ is the fundamental contravariant alternating tensor. The author recalls other definitions and other names (associate, supplement, dual) used by different writers for $*V$ or $-*V$. The paper is mainly concerned with the algebra of alternating tensors and their conjugates, with geometrical interpretations in the cases $n=2,3,4,5,6$.

H. S. Ruse (Leeds).

Tachibana, Syun-ichi. On pseudo-parallelism in Einstein spaces. Tôhoku Math. J. (2) 2, 214–219 (1951).

Falls die Klasse der Riemannschen Räume, die einem Raum C_n mit normaler konformer bezw. einem Raum P_n mit normaler projektiver Übertragung entsprechen, einen Einsteinschen Raum E_n enthalten, dann gibt es in beiden Fällen nach Sasaki [Jap. J. Math. 18, 615–622, 623–633, 791–795 (1943); diese Rev. 7, 330] und Sasaki und Yano [Tôhoku Math. J. (2) 1, 31–39 (1949); diese Rev. 11, 398] Holonomiegruppen die eine Hyperfläche zweiter Ordnung Q_n—die auch entartet sein kann—invariant lassen. Im Falle eines Raumes C_n bildet die Q_n die Sasakische Verallgemeinerung des absoluten Gebildes eines Poincaréschen Modelles. Die konformen Kreise g die Q_n in P', P'' orthogonal schneiden, sind die Geodätischen e des E_n. Im zweiten Falle wird durch eine geeignete Abwicklung des Raumes auf einen Tangentialraum eine Geodätische zu einer Geraden h, die das absolute Gebilde in P', P'' schneidet. Die Cayley-Kleinsche Metrik ergibt dann die Bogenlänge der entsprechenden Geodätischen e. Sind e und $\bar{e}$ zwei Geodätische des E_n die durch einen Punkt P_0 bezw. $\bar{P}_0$ hindurch gehen und sind P', $\bar{P}'$ die oben erwähnten Schnittpunkte der Bilder $g(h)$, $\bar{g}(\bar{h})$ in dem zu P_0 bezw. $\bar{P}_0$ gehörigen Möbiusschen (projektiven) Tangentialraum, dann heissen die im Sinne wachsender Bogenlänge gerichteten e und $\bar{e}$, $A(B)$-parallel, falls bei Abwicklung des Tangentialraumes $\bar{P}_0$ auf dem in P_0 bei Wahl einer geeigneten Verbindungskurve von P_0 und $\bar{P}_0$, P' und $\bar{P}'$ zusammenfallen. P' und $\bar{P}'$ liegen dabei in gleicher Richtung auf den Bildern von e und $\bar{e}$. Es sei g_{ih} der metrische Tensor des E_n und $P_0(x^i)$ und $\bar{P}_0(\bar{x}^i)$ mögen benachbarte Punkte sein für die (1) $\bar{x}^i=x^i+\epsilon\lambda^i$ gilt. Dabei ist λ^i ein Einheitsvektor in P_0 und ϵ eine beliebig kleine Konstante. Die Geodätischen sind dann A-parallel falls für ihre Richtungen $x^{i'}$ und $\bar{x}^{i'}$ in P_0, bezw. $\bar{P}_0$, $\pm\rho x^{i'}=\epsilon(2c)^{\frac{1}{2}}\lambda^i\pm(\bar{x}^{i'}+\epsilon\Gamma^i_{jk}\bar{x}^{j'}\lambda^k)$ mit $\rho=1\pm(2c)^{\frac{1}{2}}\epsilon g_{jk}\lambda^k\bar{x}^{j'}$ gilt. Das Vorzeichen bedeutet dabei die Wahl entsprechender Richtungen auf den Geodätischen und c ist bis auf einen unwesentlichen konstanten Faktor die skalare Krümmung des E_n. Dieselbe Gleichung gilt für benachtbarte B-parallele Geodätische. Für benachbarte Geodätische fallen daher A- und B-Parallelismus zusammen, sodass Verf. kurz von einem $\pm$ Pseudoparallelismus spricht. Durch Elimination von ρ aus den obigen Gleichungen erhält man für die Richtungsänderung ein einziges Gleichungssystem (2) $[x'^i{}_{;j}\pm(2c)^{\frac{1}{2}}g_{ih}(x^{i'}x^{k'}-y^{ik})]\lambda^j=0$, wobei das Semikolon die kovariante Ableitung bedeutet. [Bemerkung des Ref. Es handelt sich in (2) eigentlich um eine längs (1) definierte Schar von parallelen Geodätischen und $x'^i{}_{;j}\lambda^j$ ist die kovariante Ableitung von $x^{i'}$ längs $e\lambda^i$.] Auf Grund von (2) werden zwei Einheitsvektoren v^i und v^i+dv^i in x^i bzw. x^i+dx^i pseudoparallel bezeichnet falls

$$[v^i{}_{;j}\pm(2c)^{\frac{1}{2}}g_{ih}(v^iv^k-g^{ik})]dx^j=0$$

ist. Es wird nun folgender Satz bewiesen. Falls ein Einsteinscher Raum mit nicht verschwindender skalarer Krümmung n linearunabhängige pseudoparallele Vektorfelder besitzt, dann ist der Raum von konstanter Krümmung.

O. Varga (Debrecen).

Abramov, A. A. On topological invariants of Riemannian spaces obtained by the integration of tensor fields. Doklady Akad. Nauk SSSR (N.S.) 81, 125–128 (1951). (Russian)

The author sketches a proof for the following: In a Riemannian manifold V_n consider skew symmetric tensor fields (exterior differential forms), whose components are

given by fixed analytic functions of the components $g_{\alpha\beta}$ of the metric tensor and of their derivatives; among such fields the fields constructed by Pontryagin [same Doklady **43**, 91–94 (1944); these Rev. **6**, 182] are the only ones (up to coboundaries) which are topological invariants (i.e., have their integrals over arbitrary cycles independent of the choice of the metric). One considers the change of metric $g_{\alpha\beta} \to t g_{\alpha\beta}$, $t>0$. The given tensor field can be expanded in a series of terms, each homogeneous in t under this substitution; each such term is actually a polynomial in (essentially) the $g_{\alpha\beta}$ and their derivatives. The topological invariance implies that only the term of degree 0 can have non-zero integral, and one can restrict attention to this term. By counting indices, and by symmetry arguments are shown that a term can contain only second derivatives of the $g_{\alpha\beta}$ (besides $g^{\alpha\beta}$), and must in fact be one of the Pontryagin forms. Finally it is shown that the Pontryagin forms are topological invariants by showing that their "variation" under variation of the metric is a coboundary.

H. Samelson (Ann Arbor, Mich.).

Abramov, A. A. On topological invariants of Riemannian spaces obtained by the integration of pseudo-tensor fields. Doklady Akad. Nauk SSSR (N.S.) **81**, 325–328 (1951). (Russian)

By an analysis similar to that in the note reviewed above it is shown that the only skew-symmetric pseudo-tensor fields (pseudo meaning besides the usual transformation character a change in sign if the Jacobian is negative), which is an analytic function of the metric tensor and its derivatives and which is a topological invariant (the integral of the differential form over any cycle is independent of the choice of metric), is the Allendoerfer-Weil curvature form [Trans. Amer. Math. Soc. **53**, 101–129 (1943); these Rev. **4**, 169]. (Together with those found in the note reviewed above this is then exactly the characteristic ring.)

H. Samelson (Ann Arbor, Mich.).

Ehresmann, Charles. Les prolongements d'une variété différentiable. V. Covariants différentiels et prolongements d'une structure infinitésimale. C. R. Acad. Sci. Paris **234**, 1424–1425 (1952).

[This note continues five others: same C. R. **233**, 598–600, 777–779, 1081–1083 (1951); **234**, 587–589, 1028–1030 (1952); these Rev. **13**, 386, 584, 780.] The author continues his study of the spaces formed by the contact elements of higher order of derivation in a manifold. He defines a number of concepts: covariant differential in prolongements of a manifold, covariant differential with respect to an infinitesimal structure (cross-section in a prolongement). Special and general affine connections of order k. These are extensions of classical notions (for $k=1$). *H. Samelson.*

Ehresmann, Charles. Sur la théorie des variétés feuilletées. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **10**, 64–82 (1951).

The theory of leaf spaces (variétés feuilletées), developed in this paper, has the purpose of laying the topological foundations of a global theory of completely integrable Pfaffian systems on a differentiable manifold V_n. A differentiable manifold is called leaved when such a Pfaffian system is given and the complete integral varieties are called the leaves. The situation bears close resemblance to that of fiber spaces and many of the notions pertaining to the latter are carried over. In particular, from a covering homotopy theorem it follows that if a leaf is contractible in V_n it is parallelisable. The author also considers transversal submanifolds of a Pfaffian system, which are submanifolds having the property that at every point its tangent vector space is supplementary to the tangent vector space associated to the point by the given Pfaffian system. If the field of elements of contact completely orthogonal to the field defined by the Pfaffian system (assuming a Riemann metric in V_n) is orientable and if the cycle carried by a transversal submanifold V_p has an integral multiple which is homologous to zero in V_n, the Euler-Poincaré characteristic of V_p is zero. Moreover, the Euler-Poincaré characteristics of homologous transversal submanifolds are equal. *S. Chern.*

Kosambi, D. D. Path-spaces admitting collineations. Quart. J. Math., Oxford Ser. (2) **3**, 1–11 (1952).

Es wird untersucht wann der Raum der Bahnen (1) $\ddot{x}^i+\alpha^i(x,\dot{x})=0$, $\dot{x}^i=dx^i/dt$, $\ddot{x}^i=d\dot{x}^i/dt$ $(i=1,2,\cdots,n)$ eine kontinuierliche Transformationsgruppe des Urraumes x^i in sich zulässt. Die vom Verf. bewiesenen Sätze beziehen sich teils auf Gruppen für die weder eine formale Reihenentwicklung der Gruppe noch der α^i vorhanden ist, teils auf solche für die dies zutrifft und α^i von der Form

$$\alpha^i=A^i+A^i_j\dot{x}^j+\Gamma^i_{jk}\dot{x}^j\dot{x}^k+A^i_{jkl}\dot{x}^j\dot{x}^k\dot{x}^l+\cdots$$

ist. In die erste Kategorie gehören u.a. folgende Ergebnisse. Der Raum gestattet eine Translationsgruppe T_n bezw. eine Untergruppe der homogenen Linearengruppe falls α^i von der Form $\alpha^i=\alpha^i(\dot{x})$ bezw. $\alpha^i=\dot{x}^iH(x,\dot{x})+x^iI(x,\dot{x})$ ist. H und I sind dabei absolute Invarianten der ersten Erweiterung der fraglichen Gruppe. In die zweite Kategorie gehört folgender Satz. Der allgemeinste Raum der eine Liesche Gruppe zulässt, hat ein α^i der Gestalt $\alpha^i=\Gamma^i_{jk}\dot{x}^j\dot{x}^k+\lambda^i$. Dabei gestattet der Affinzusammenhang Γ^i_{jk} und der kontravariante Vektor λ^i die gegebene Gruppe. Verf. leitet auch einige Ergebnisse über projektiv-ebene Räume her. Methodisch verwendet Verf. die zu (1) gehörigen Variationsgleichungen und die verallgemeinerten Lieschen Operation $\mathfrak{L}$. Bei der Behandlung der Integrabilitätsbedingungen dieser Variationsgleichungen ist es dabei wesentlich, dass $\mathfrak{L}$ mit allen Differentialoperatoren vertauschbar ist. *O. Varga.*

Vagner, V. V. The algebraic theory of differential groups. Doklady Akad. Nauk SSSR (N.S.) **80**, 845–848 (1951). (Russian)

The theory of geometrical differential objects can in the main be reduced to the theory of the representations of the differential group $\mathfrak{D}_{(v,n)}$, which is the transformation group of all differentials of order v inclusive of n variables ξ^α. In this paper this group is studied from a purely algebraic point of view by the introduction of the algebra $Q_{(v,n)}$ on the field of real numbers of all polynomials of the form $\sum_{i=1}^{v} c_{\alpha_1\cdots\alpha_i} x^{\alpha_1}\cdots x^{\alpha_i}$, $\alpha=1,\cdots,n$, where the formal multiplication is associative, commutative and satisfies the condition that the products $x^{\alpha_1}\cdots x^{\alpha_i}$ form a basis; $x^{\alpha_1}\cdots x^{\alpha_{v+1}}=0$. The powers Q^i, $i=1,\cdots,v+1$, give a decreasing succession of ideals, $Q^{v+1}=0$ denotes the nilpotent $Q_{(v,n)}$. A homomorphic mapping θ of $Q_{(v,n)}$ on $Q_{(v,m)}$ is given by n polynomials in $Q_{(v,m)}$:

$$\theta(x^\alpha)=\sum_{i=1}^{v} b^\alpha_{\alpha_1\cdots\alpha_i} y^{\alpha_1}\cdots y^{\alpha_i}, \quad i=1,\cdots,m.$$

Similarly,

$$\chi(y^\alpha)=\sum_{i=1}^{v} c^\alpha_{\rho_1\cdots\rho_i} z^{\rho_1}\cdots z^{\rho_i}, \quad \rho=1,\cdots,l,$$

represents a homomorphic mapping χ of $Q_{(v,m)}$ on $Q_{(v,n)}$. An analysis of these mappings shows that the group of all

automorphisms of $Q_{(r,n)}$ is isomorphic with the group $\mathfrak{D}_{(r,n)}$. Several consequences are drawn from this result. It is also observed that, if by $\Theta_{(r,n,m)}$ is denoted the manifold of all homomorphic mappings of $Q_{(r,n)}$ on $Q_{(r,m)}$, then the product $\mathfrak{D}_{(r,n)}\times\mathfrak{D}_{(r,m)}$ can be taken as the group of transformations of $\Theta_{(r,n,m)}$. *D. J. Struik* (Cambridge, Mass.).

Yastrebov, Yu. N. Generalized group spaces. Doklady Akad. Nauk SSSR (N.S.) **81**, 161–164 (1951). (Russian)

It is known that it is possible to determine, in a space of a continuous Lie group, two affine connections of curvature zero, I and II, invariant under the group, and with the same geodesics. Moreover, if a and b are two geodesics parallel with respect to connection I (or II) and the geodesic m intersects a and b, then every geodesic l parallel to m with respect to II (or I) which intersects a, also intersects b. In this paper it is shown that there are pairs of affine connections satisfying these properties which do not belong to groups. The pair given by

$$\overset{1}{\Gamma}{}^{\alpha}_{\mu\nu}=u^{\alpha}(-a_{\sigma\rho}a_{\omega\nu}u^{\sigma}u^{\omega}+a_{\mu\nu})-\delta_{\nu}{}^{\alpha}a_{\sigma\rho}u^{\sigma}$$

$$\overset{2}{\Gamma}{}^{\alpha}_{\mu\nu}=u^{\alpha}(-a_{\sigma\rho}a_{\omega\nu}u^{\sigma}u^{\omega}-a_{\mu\nu})+\delta_{\nu}{}^{\alpha}a_{\sigma\rho}u^{\sigma},$$

where $a_{ij}=-a_{ji}$, constitutes for $n\geqq 4$ a unique such class of manifolds. For $n=3$ the uniqueness is still in question. There is a special discussion of the subprojective case.

D. J. Struik (Cambridge, Mass.).

Širokov, A. P. A gonometric system in the geometry of Finsler. Trudy Sem. Vektor. Tenzor. Analizu **8**, 414–424 (1950). (Russian)

A bimetric system is a three-dimensional manifold (x_1, x_2, x_3) in which three forms $\omega^k=\omega_s{}^k dx^s$ $(k=1, 0, 2)$ are given, determined but for the transformations

$$\tilde{\omega}^1=\omega^1+\lambda\omega^0,\quad \tilde{\omega}^0=\nu\omega^0,\quad \tilde{\omega}^2=\omega^2+\mu\omega^0$$

[see P. Raševskiĭ, same Trudy **5**, 21–147 (1941); these Rev. **8**, 231]. Such systems have importance for the study of metrical dualities. The ω satisfy the equations of structure and can be written in the form

$$\omega^1=Adx+Bdz+\lambda(dy-zdx),$$
$$\omega^0=\nu(dy-zdx),$$
$$\omega^2=Cdx+Ddz+\mu(dy-zdx),$$

where A, D are functions of x, y, z, and B, C functions of x, z, ν. Here $\omega^0=0$ gives curves, $\omega^1=\omega^2=0$ gives null lines in the first metric, $\omega^2=\omega^0=0$ null lines in the second metric. The length of a curve in the first metric is $\int\omega^1$, in the second metric $\int\omega^2$. In such a way a bimetric geometry can be built up in which such geometries as that of Finsler can be included. For a gonometric system the null lines of the first metric are points, those of the second metric are curves of a given family. Angle is obtained by taking ω^2 along the point, arc length by taking ω^1 along the curve [B.V. Lesovoĭ, ibid. **6** (1948)]. For a gonometric system in a Finsler plane the ω^k are derived, and conclusions drawn. The paper ends with some remarks on the introduction into Finsler geometry of bigonometric and bilongometric systems, connected with two families of curves. *D. J. Struik.*

Moór, Arthur. Quelques remarques sur la généralisation du scalaire de courbure et du scalaire principal. Canadian J. Math. **4**, 189–197 (1952).

In a 2-dimensional Finsler space the curvature tensor contracted by l^s, i.e. $R_0{}^i{}_{jk}(=R_s{}^i{}_{jk}l^s)$, and the torsion tensor A_{ijk} are expressed by

$$R_0{}^i{}_{jk}=\Re h^i(l_jh_k-l_kh_j),\quad A_{ijk}=2\Im h_ih_jh_k,$$

using the curvature scalar $\Re$ and the principal scalar $\Im$ introduced by Berwald [J. Reine Angew. Math. **156**, 191–210, 211–222 (1927); Ann. of Math. (2) **42**, 84–112 (1941); these Rev. **2**, 304], where l^i and h^i are the unit vector in the direction of the supporting element (x, x') and the unit normal vector respectively. The author derives these two scalars by the same expressions in an n-dimensional Finsler space, making use of the unit vector in the direction of the Euler vector along a curve in place of h^i, and he states a necessary and sufficient condition for consistency of the same relations as those above in the case of 3 dimensions. At the end he remarks that the considered invariants and tensors characterize only along a curve, but not along a curve when the fundamental element of the space is the linear element of second order (x, x', x'') as in a Kawaguchi space. *A. Kawaguchi* (Sapporo).

Payne, W. T. Elementary spinor theory. Amer. J. Phys. **20**, 253–262 (1952).

NUMERICAL AND GRAPHICAL METHODS

Chandrasekhar, S., Elbert, Donna, and Franklin, Ann. The X- and Y-functions for isotropic scattering. I. Astrophys. J. **115**, 244–268 (1952).

Chandrasekhar, S., and Elbert, Donna. The X- and Y-functions for isotropic scattering. II. Astrophys. J. **115**, 269–278 (1952).

The functions in question are solutions of the integral equations

$$X(\mu)=1+\mu\omega_0\int_0^1(\mu+\mu')^{-1}[X(\mu)X(\mu')-Y(\mu)Y(\mu')]d\mu'$$

$$Y(\mu)=e^{-\tau/\mu}+\mu\omega_0\int_0^1(\mu-\mu')^{-1}[Y(\mu)X(\mu')-X(\mu)Y(\mu')]d\mu'.$$

Approximations were described in Chandrasekhar's book Radiative Transfer, chapter VIII, §60 [Oxford Univ. Press, 1950; these Rev. **13**, 136]. In the first of these papers the approximate solutions were improved by iteration and are tabulated for $0\leqq\mu\leqq 1$ for the following values of the parameters: $\omega_0=.95, .9, .8, .5$, $\tau=.05$ (.05) .25, .5, 1. The tables are to 4 or 5D, and the difference between the tabulated solution and the corrected second approximation (of Chandrasekhar's book) is also given.

In the second paper, "standard solutions" are defined in the case $\omega_0=1$. They satisfy the relations $\int_0^1 X_s(\mu)d\mu=2$, $\int_0^1 Y_s(\mu)d\mu$. The moments of X, Y, X_s, Y_s are called α_n, β_n, $\alpha_n{}^s$, $\beta_n{}^s$. The laws of diffuse reflection are expressed in terms of the solutions

$$X^*(\mu)=X_s(\mu)+Q\mu[X_s(\mu)+Y_s(\mu)],$$
$$Y^*(\mu)=Y_s(\mu)-Q\mu[X_s(\mu)+Y_s(\mu)],$$

where

$$Q=[\alpha_1{}^s-\beta_1{}^s]/[(\alpha_1+\beta_1)\tau+2(\alpha_2+\beta_2)].$$

Table 1a gives α, β_0, α_0, α_1, β_1 and

$$\delta=1-\{(2-\omega_0\alpha_0)\alpha_1+\omega_0\beta_0\beta_1\}$$

for $\tau=.05$ (.05) .25, .5, 1 and $\omega_0=.95, .90, .80, .5$. Table 1b gives $\alpha_0+\beta_0$, $\alpha_1+\beta_1$, $\alpha_2+\beta_2$, $\alpha_1{}^s$, $\beta_1{}^s$, $(\alpha_1{}^s)^2-(\beta_1{}^s)^2$, Q, and δ

for the same values of τ and $\omega_0=1$. Table 2 gives X^*, Y^*, X_s, Y_s for $\omega_0=1$, $\tau=.05$ (.05) .25, .5, 1, $\mu=0(.01)1$. Table 3 contains certain combinations of the moments.

A. Erdélyi (Pasadena, Calif.).

Umeda, Kwai. Über die Thomas-Fermi-Dirac-Funktionen und ihre Tabellierung. J. Fac. Sci. Hokkaido Univ. Ser. II. 3, 171–244 (1942); Ergänzungen und Berichtigungen: 245 (1949).

The function $\psi(x)$ satisfies the differential equation

$$\frac{d^2\psi}{dx^2}=x[\beta+(\psi/x)^{1/2}]^2,$$

$\psi(0)=1$, and the first zero of $\psi(x)$ is a double zero. Table 1 (pp. 184–197) gives 4S tables of ψ and ψ' (each multiplied by a constant factor) for 28 different cases, and Table 2 (pp. 198–244) gives tables for all the elements from H ($z=1$) to U ($Z=92$). *A. Erdélyi* (Pasadena, Calif.).

***Willers, Fr. A. Mathematische Maschinen und Instrumente.** Akademie-Verlag, Berlin, 1951. xii+318. DM 34.21.

This is a considerably revised version of the author's Mathematische Instrumente [Oldenbourg, München-Berlin, 1943]; there is naturally more emphasis on (digital) machines. The chapter on instruments for the solution of algebraic equations and of systems of linear equations has been omitted. The author has taken account of the annotations (about developments in Russia) made in a Russian translation which appeared in 1949. The book is noteworthy for its 251 photographs and diagrams and for a bibliography of 871 items.

There are chapters dealing with desk calculators (revealing the mechanisms of European machines) and on high speed automatic digital computing machines. The section on "smaller" instruments is valuable in so far as it indicates the considerable variety of special and general purpose instruments available commercially. This section includes chapters entitled: drawing of curves and measurements on curves (considerably revised from the 1943 account), planimeters, harmonic analysers and Stieltjes planimeters, integraphs and integrators. The concluding chapter gives an account of differential analysers; it has been revised for this edition, but there is little mention of electronic differential analysers. *J. Todd* (Washington, D. C.).

Jarden, Dov. On the division of great numbers by great numbers. Riveon Lematematika 5, 59–61 (1952). (Hebrew)

Methods of long division, convenient for pencil and squared paper computation and checking.

E. G. Straus (Los Angeles, Calif.).

Banachiewicz, Tadeusz. Sur l'interpolation dans le cas des intervalles inégaux. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. 42 (1949), 9–11 (1952). (French. Polish summary)

Banachiewicz, Tadeusz. A general least squares interpolation formula. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. 42 (1949), 218–219 (1952). (English. Polish summary)

Unger, Heinz. Lagrange-Hermitesche Interpolation im Komplexen. Z. Angew. Math. Physik 3, 51–65 (1952).

To set up an interpolation polynomial $P(z)$ for an analytic function $g(z)$ in the complex plane, the author starts from an extension of the Lagrange formula given by Hermite [Oeuvres, vol. 3, Gauthier-Villars, Paris, 1912, pp. 432–443]

$$P(z)=\sum_{\mu=1}^{n}\sum_{\rho=0}^{q}L_{\mu,\rho}(z)g^{(\rho)}(z_\mu),$$

where the values of $g(z)$ and its first q derivatives are known at the interpolation points z_μ. When these points are chosen at the vertices of a regular polygon of n sides, it is shown that

$$L_{\mu,\rho}(z)=e^{i(2\pi/n)\rho(\mu-1)}L_{1,\rho}[z^{(\mu)}],\quad z^{(\mu)}=ze^{i(2\pi/n)(1-\mu)}.$$

Thus for a square grid and $P(z)$ of degree 11 it is necessary only to tabulate $L_{1,0}$, $L_{1,1}$, $L_{1,2}$. The remainder term is discussed and the consequences of adding the centre of the polygon as an interpolation point.

L. M. Milne-Thomson (Greenwich).

***Faddeeva, V. N. Vyčislitel'nye metody lineĭnoĭ algebry. [Computational methods of linear algebra].** Gosudarstv. Izdat. Tehn.-Teoret. Lit., Moscow-Leningrad, 1950. 240 pp. 9.85 rubles.

This is a textbook on numerical methods for solving finite systems of linear equations, inverting matrices, and calculating the eigenvalues of finite matrices, all with desk calculators. Although the book is far from exhaustive, the mathematical elegance, the breadth of material, and the number of error-free numerical examples make this by far the finest book to appear in the field.

In the first chapter of 67 pages the author develops necessary background of linear algebra and analysis. The reader is presumed to know elementary determinant theory and Cramer's rule, but nothing about matrices. Confining herself to the complex field, the author leads us quickly through the addition and multiplication of matrices, inverse matrices, the Cayley-Hamilton theorem, similarity of matrices, the triangular decomposition theorem, n-dimensional vector spaces, linear dependence, inner products, biorthogonal systems, linear transformations, characteristic vectors, and quadratic forms. Surprisingly few proofs are omitted, but space permits little motivation of the material. The Jordan canonical form and elementary divisors are presented without proof. At the end of the chapter we learn about the pth power norm of vectors, for $p=1, 2, \infty$. The corresponding three matrix norms $\|A\|$ ($=\max\|Ax\|_p$ over the unit sphere $\|x\|_p=1$) are introduced and are used in several theorems concerning the convergence of $\sum A^m$ to $(E-A)^{-1}$, $E=$unit matrix. This analytical material, though less frequently expounded than the algebra of matrices, is quite as essential in practical matrix work.

The second chapter of 79 pages is devoted to the solution of linear systems. We learn of Gauss's elimination method in expanded and compact forms and its interpretation as the triangular resolution of a matrix. The square-root method is recommended as one of the most effective for solving symmetric systems. "Below-the-line" devices are recommended for avoiding the back solution in Gauss's method. Partitioning and escalator methods are demonstrated for elimination by blocks. The rest of the chapter (27 pages) is devoted to two classical iterative methods: "ordinary iteration" (the Gesamtschrittverfahren), and "Seidel's method" (the einzelschrittverfahren or "Gauss-Seidel method," apparently advocated by neither Gauss nor Seidel!). Convergence criteria for these two methods are dealt with carefully. We even find Reich's 1949 proof [Ann. Math. Statistics 20, 448–451 (1949); these Rev. 11, 136] that, for a symmetric matrix, Seidel's method converges if and only if A is positive defi-

nite. Each method in the chapter is illustrated by a numerical example, usually of order 4. These examples are practically free of errors or misprints.

The third chapter of 92 pages contains a number of ways of determining eigenvectors and eigenvalues. We find the methods of A. N. Krylov, Danilevskiĭ, Samuelson, Leverrier, and Faddeev-Sominskiĭ for determining the characteristic polynomial. The last was independently discovered by Frame [abstract, Bull. Amer. Math. Soc. 55, 1045 (1949)] and published by Dwyer [Linear Computations, Wiley, New York, 1951; these Rev. 13, 283]. Danilevskiĭ's seems to be the author's preferred method. The escalator and interpolation methods of getting all eigenvalues are presented. The standard iterative process (power method) is given for approximating the dominant eigenvalue and eigenvector. Aitken's δ^2 process for accelerating convergence is described. The following eigenvalues are found by λ-differencing methods, and there are discussions of the two situations where one or two eigenvalues dominate. Matrices with nonlinear divisors are discussed briefly. There are numerical examples of all the methods. At the end the λ-differencing idea is applied to the acceleration of the iterative solution of linear equations. There is a final apology for not having been able to include the gradient (least-squares) methods for solving linear equations, with references to the work of Kantorovič and others on these methods.

There is a brief bibliography of 23 Russian and 13 non-Russian titles, but no index. The typesetting is good and the paper of relatively good quality. There are no exercises for the reader.

Faddeeva probably errs in stating (p. 65) that one can use a continuity argument to extend to matrices with nonlinear elementary divisors the proof in the case of linear divisors that, if all $|\lambda_i(A)|<1$, $A^k\to 0$. The proof of the "only if" part of Reich's theorem (cited above) can be shortened from 3 pages to a few lines, as Ostrowski has discovered [unpublished]. One wishes the author had said something about the effect of automatic computing machinery on computing methods. There are a number of minor misprints in formulas.

The National Bureau of Standards has issued a translation of the first chapter of Faddeeva's book [N. B. S. Report 1644 (1952)], and a translation of the rest is in progress. *G. E. Forsythe* (Los Angeles, Calif.).

Hertwig, August. Die Zahlenrechnungen bei der Lösung zahlreicher linearer Gleichungen. Mitt. Deutsch. Akad. Luftfahrtforschung 2, 341–369 (1943).

Brock, P., and Murray, F. J. The use of exponential sums in step by step integration. Math. Tables and Other Aids to Computation 6, 63–78 (1952).

The authors derive open quadrature formulas of the type

$$(1)\quad \int_t^{t+h} F(t)\,dt = h[a_0F(t)+a_1F(t-h)+\cdots+a_{n-1}F(t-(n-1)h)]$$

in which the n coefficients $a_0, \cdots, a_{n-1}$ are determined so as to make (1) exact if $F(t)=e^{\nu_i t}$, $i=1, \cdots, n$. A corresponding set of closed quadrature formulas are derived to satisfy similar conditions. The open formulas can be used to predict, and the corresponding closed formula to correct, a new value of the dependent variable in the step-by-step integration of a first-order differential equation, just as is done with the familiar Newton-Cotes open and closed pairs of quadrature formulas. The error of (1) is studied and is shown to be $O(h^{n+1})$. The use contemplated for formulas of this type is the solution of a system of equations $z_i'=f_i(z_1, \cdots, z_n, t)$ in which it is known that the right-hand members, when obtained as functions of time, are approximately exponential sums $\sum c_j e^{\lambda_j t}$. The λ_j are not initially known. However one may have initial estimates of sufficient accuracy to aid in the selection of the ν_i. *W. E. Milne* (Corvallis, Ore.).

Curtiss, C. F., and Hirschfelder, J. O. Integration of stiff equations. Proc. Nat. Acad. Sci. U. S. A. 38, 235–243 (1952).

By a "stiff" equation the authors mean an equation $y'=f(x, y)$ for which there is a critical integral curve such that integral curves on either side of the critical curve turn sharply away from it as x increases (or decreases). In such a situation it is obviously difficult to find this integral curve by step-by-step numerical integration, since any small error causes a shift to a new curve diverging sharply from the one desired. The authors present a scheme of successive substitutions which singles out the desired solution.

W. E. Milne (Corvallis, Ore.).

Weissinger, Johannes. Eine Fehlerabschätzung für die Verfahren von Adams und Störmer. Z. Angew. Math. Mech. 32, 62–67 (1952).

In an earlier paper [same Z. 30, 356–363 (1950); these Rev. 13, 286] the author presented an improvement in von Mises' estimate of the error in Adams' method of extrapolation for the numerical solution of $y'=f(x, y)$. The present paper extends the idea to cover Adams' method of interpolation for $y'=f(x, y)$ and Störmer's method for $y''=f(x, y)$.

W. E. Milne (Corvallis, Ore.).

***Mikeladze, Š. E. Novye metody integrirovaniya differencial'nyh uravneniĭ i ih priloženiya k zadačam teorii uprugosti. [New methods of integration of differential equations and their application to problems in the theory of elasticity.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951. 291 pp. 11.50 rubles.

This book gives numerical methods for solution of buckling and vibration problems involving columns and shafts. The principal mathematical interest lies in the first chapter on boundary problems with linear differential equations. The author sets forth here several numerical methods of finding characteristic values, most of them taken from his own writings. These methods employ numerical formulas for integration and differentiation, many of them novel or at least uncommonly used, to construct an equation for the characteristic values of the system. In some cases, the system is converted to an integral equation, in others formulas for derivatives are used directly. Free use of unbalanced formulas is made. The chapters after the first are devoted to specific applications. The second chapter deals with buckling of columns with various types of end support and lateral restraint. No assumption as to uniformity of the cross-section is generally required. The four remaining chapters are much shorter, and deal with stability under loads distributed longitudinally, stability of curved beams, longitudinal vibrations of bars, and transverse vibrations of bars and strings.

R. E. Gaskell (Seattle, Wash.).

Meyerott, R. E., Luke, P. J., Clendenin, W. W., and Geltman, S. A numerical variational method. Physical Rev. (2) 85, 393–400 (1952).

Für das Eigenwertproblem

$$\psi''+[\lambda+U(x)]\psi=0, \quad \psi(a)=\psi(b)=0$$

wird eine Näherungsfunktion in folgender Weise entwickelt: Man teilt das Intervall in n gleiche Teile von der Länge h. Sowohl die gegebene Funktion $U(x)$ als auch die gesuchte Funktion $\psi(x)$ wird zwischen den einzelnen Teilungspunkten (x_n, x_{n+1}) linear interpoliert, sodass als Unbekannte für das zugehörige Minimalproblem die Ordinaten in den Teilungspunkten erscheinen. Durch Nullsetzung der ersten Variation erhält man eine Differenzengleichung

$$\tfrac{1}{6}[1+h^2(\lambda+U)](\delta^2\varphi)_n+h^2(\lambda+U)\varphi_n=0.$$

Hierauf folgt eine genauere Diskussion der erzielten Näherung in Verbindung mit einem Hinweis auf eine Arbeit von D. R. Hartree [Mem. Proc. Manchester Lit. Philos. Soc. 77, 91–107 (1933)] und ferner eine Methode, bei der das mittlere Fehlerquadrat möglichst klein gemacht wird. Das Verfahren wird dann auf analoge zweidimensionale Probleme übertragen. Die Interpolation in den einzelnen Rechtecken erfolgt so, dass innerhalb des Rechteckes Linearität sowohl in x als auch in y herrscht und dass ferner die Näherungsfunktion stetig ist. *P. Funk* (Wien).

Milne, W. E. Numerical determination of characteristic numbers. J. Research Nat. Bur. Standards 45, 245–254 (1950).

Als Erläuterung für die leicht verallgemeinerungsfähige Methode dient das Eigenwertproblem $L[u]+\lambda^2 u=0$, wobei der Verf. sich den linearen Differentialausdruck normiert denkt in der Form $L[u]=u''+p(x)u'+q(x)u$. Das Problem

$$\frac{\partial^2 V}{\partial t^2}=L[u], \quad V(a,t)=V(b,t)=0,$$

$$V(x,0)=F(x), \quad \left.\frac{\partial V}{\partial t}\right|_{t=0}=0,$$

wird umgewandelt in das entsprechende Problem für Differenzengleichungen, wobei $F(x)$ durch ein Wertesystem, bei dem $V_{10}\neq 0$ und $V_{i0}\equiv 0$ für $i>1$ und die zweite Anfangsbedingung durch die Gleichung $V_{i1}=V_{i-1}$ ersetzt wird ($V_{ij}=V(a+ih, jh)$). Andererseits wird entsprechend dem Bernoullischen Ansatz für die V_{ij} ein Ansatz gemacht $V_{ij}=\sum c_k u_{ki}\cos\lambda_k h_j$ und hieraus eine Gleichung von der Form

$$\cos n\mu+A_{n-1}\cos(n-1)\mu+\cdots+A_1\cos\mu+A_0=0$$

gewonnen, wobei die Wurzeln $\mu_k=\lambda_k h$ sind und die A_i sich aus der linearen Abhängigkeit der $n$ten Zeile der Matrix (V_{ij}) (Dreiecksmatrix) von den $(n-1)$sten Zeilen ergeben. Es folgen einige numerische Illustrationsbeispiele.

P. Funk (Wien).

Please see the corresponding note on p. 1140

Stiefel, Eduard. Über einige Methoden der Relaxationsrechnung. Z. Angew. Math. Physik 3, 1–33 (1952).

This is a concise up-to-date exposition of the relaxation process of Southwell and his followers as applied to the iteration methods of Seidel, Hardy Cross, etc., for the solution of linear equations arising in the numerical solution of boundary value problems for linear partial differential equations. The paper starts with the necessary mathematical foundations, the cornerstone being the quadratic form associated with the linear equations. Next the author describes the relaxation process in a comprehensive manner, states and proves basic theorems, and illustrates graphically the situation where $n=2$. Then he takes up block relaxation and other devices for overall improvement of groups of values, and also presents several ways to improve the block technique. The method of steepest descent is explained as is also the n-step process of Hestenes, Lanczos, etc., leading to a solution in a finite number of steps. The paper concludes with a discussion of harmonic weight functions and estimates for the magnitude of the error. *W. E. Milne.*

Nygaard, Kristen. On the solution of integral equations by Monte-Carlo methods. Norwegian Defence Research Establishment, Rep. no. **F-R 94**, 9 pp. (1952).

The equations are of the form

$$H(x)=f(x)+\kappa\int_a^b H(\xi)K(\xi,x)d\xi,$$

where K, f, and κ are positive, $\int_a^b f(x)dx=1$, $\int_a^b K(\xi,x)dx\leqq 1$, $\kappa\leqq 1$. Divide the interval (a,b) into 10^p subintervals (x_n, x_{n+1}) with a "representative abscissa" ξ_n on each, satisfying $\int_{x_n}^{\xi_n} f(x)dx=\int_{\xi_n}^{x_{n+1}} f(x)dx=10^{-p}/2$. Select $x_0=\xi_n$ by a p-digit "random number". For x_1, make subdivisions based on $K(x_0,x)$, select x_1 as a representative abscissa, or else terminate the process if the q-digit random number $R_2\geqq 10^q\int_a^b K(x_0,x)dx$. Terminate the process if an r-digit random number $R_3>\kappa\cdot 10^r$; otherwise subdivide by $K(x_1,x),\cdots$. If from N sequences, on any interval (x', x'') there are C points having successors, then C/N is an estimate of $\int_{x'}^{x''} H(x)dx$. The same sequences can be used for the equation on a subinterval (a', b') of (a, b). Extension is made to the complex case. *A. S. Householder* (Oak Ridge, Tenn.).

Sykes, J. B. Approximate integration of the equation of transfer. Monthly Not. Roy. Astr. Soc. 111, 377–386 (1951).

The standard method of solving the equations of transfer by replacing integrals by sums [cf. S. Chandrasekhar, Radiative Transfer, Oxford, 1950; these Rev. 13, 136] is applied to the solution of the simplest transfer problem (namely, that of conservative isotropic scattering in semi-infinite plane parallel atmospheres) by using a quadrature formula based on $d^n[\mu(1-\mu)]^n/d\mu^n$ $(0<\mu<1)$ instead of, for example, Gauss' formula which is based on the Legendre polynomials. It is shown that the lower orders of approximation derived on the basis of the new quadrature formula are somewhat better than the corresponding ones derived on Gauss' or Newton's formula. *S. Chandrasekhar.*

Yuškov, P. P. On the correction of the coefficients obtained in the usual practical harmonic analysis. Akad. Nauk SSSR. Inženernyĭ Sbornik 10, 213–222 (1951). (Russian)

In an earlier paper [same Sbornik 6, 197–210 (1950); these Rev. 13, 288] the author obtained approximations a_m and b_m to the Fourier coefficients A_m and B_m for a function $f(x)$ by means of formulas

$$(1) \qquad a_m=c_m\alpha_m, \quad b_m=c_m\beta_m,$$

in which α_m and β_m are the coefficients found by the usual process of harmonic analysis with n subdivisions of the period. In the present paper the idea is extended so as to express a_m as a linear combination of α's and b_m as a linear combination of β's. Typical formulas are

$$a_1=1.00006\alpha_1-0.06585\alpha_7-0.19901\alpha_9,$$
$$a_2=1.00079\alpha_2-0.02670\alpha_8-0.2765\alpha_{10}, \text{ etc.}$$

Tables of the coefficients are supplied for four different cases.

W. E. Milne (Corvallis, Ore.).

Hunter, G. Truman. Modern computing machines. J. Franklin Inst. **253**, 567–583 (1952).

Bückner, Hans. Le grandi macchine calcolatrici. Ricerca Sci. **21**, 1316–1363 (1951).

Mitchell, K. Use of the proposed electric isograph. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2411** (9014), 24 pp. (1951).

Dadda, Luigi. Ricerche sul metodo del modello elettrolitico. Ricerca Sci. **22**, 685–693 (1952).

Harmuth, Henning. Eine elektronische Rechenmaschine für statistische Berechnungen. Statist. Vierteljschr. **5**, 32–38 (1952).

Goedkoop, J. A. Computation of crystal structures with electronic analogy machines. Nederl. Tijdschr. Natuurkunde **18**, 99–110 (1952). (Dutch)

Schoone, J. C. Machines for carrying out Fourier summations in the computation of crystal structures. Nederl. Tijdschr. Natuurkunde **18**, 85–98 (1952). (Dutch)

ASTRONOMY

Mineo, Massimo. Coordinate geografiche sulle superficie con applicazioni alla geodesia. Matematiche, Catania **6**, 126–131 (1951).

Elementary methods are used to establish the differential correlation between geographic coordinates and others such as astronomical geodetic and astronomical gravimetric. The relations set forth are well known in geodesy.

N. A. Hall (Minneapolis, Minn.).

Eichhorn, Heinrich. Die Ausnahmefälle bei der Bestimmung einer Kreisbahn. Anz. Öster. Akad. Wiss. Math.-Nat. Kl. **1951**, 228–235 (1951).

Eichhorn, Heinrich. Zur Erfassung der Ausnahmefälle bei der Bahnbestimmung. Anz. Öster. Akad. Wiss. Math.-Nat. Kl. **1951**, 235–241 (1951).

An earlier article by the same author [Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **158**, 203–225 (1950); these Rev. **12**, 753] deals with the development of the Jacobian determinant that vanishes if the equations for determining an elliptic orbit are indeterminate. In the first of the two papers under review the case of a circular orbit is examined. The four equations are of a simpler character than the six equations of the general case. On account of this simplification the discussion can be carried through completely, and a geometrical interpretation of the conditions of indeterminateness succeeds. In the second paper the more general problem is reconsidered from the point of view of numerical applications. By elimination of some of the unknowns a reduced system of equations may be obtained; the reduced system contains some of the zero points of the complete Jacobian, and the laborious computation of the complete determinant is avoided. An example of the method for the circular case is given. *D. Brouwer* (New Haven, Conn.).

Rudkjøbing, Mogens. On the internal constitution of relativistically degenerate stars. Danske Vid. Selsk. Mat.-Fys. Medd. **27**, no. 5, 13 pp. (1952).

The author shows that by including the "spin-orbit" interaction in addition to the relativistic mass variation with velocity, the equation governing the hydrostatic equilibrium of a completely degenerate stellar configuration can be reduced to

$$\frac{1}{\eta^2}\frac{d}{d\eta}\left(\eta^2\frac{d\phi}{d\eta}\right)=-\left[\phi^2-\left(\eta\frac{d\phi}{d\eta}\right)^2-\frac{1}{y_c^2}\right]^{3/2}.$$

This equation differs from the usual one [cf. S. Chandrasekhar, An introduction to the study of stellar structure, Univ. Chicago Press, 1939, chap. 11] in the appearance of the term in $\eta\, d\phi/d\eta$ on the right-hand side. By integrating the foregoing equation for values for y_c^2 (which is essentially the central density in suitable units) equal to 0.5, 0.2 and 0.1, the author shows that the essential modification which the spin-orbit interaction introduces is to decrease the value of the critical mass in the usual theory by about 15%.

S. Chandrasekhar (Williams Bay, Wis.).

Langebartel, Ray G. On the motion in barred spirals. Stockholms Observatoriums Annaler **17**, no. 3, 11 pp. (1 plate) (1951).

It is assumed that in a homogeneous oblate spheroid ($r^2/a^2+z^2/b^2=1$) a "bar" develops which can be described as a perturbation ($\delta\rho$) in the density of the form

$$\rho^{-1}\delta\rho=\epsilon\gamma^4r^2a^{-2}\cos 2\theta,$$

where γ is a constant and θ is the polar angle in the (x, y)-plane. The corresponding perturbations in the external and the internal gravitational potentials are explicitly evaluated and the resulting character of the changes which will initially take place in orbits which are originally circular in the (x, y)-plane are described. The results of this analysis are related to Lindblad's theory of spiral structure in nebulae. *S. Chandrasekhar* (Williams Bay, Wis.).

Bondi, C. M., and Bondi, H. Models for red giant stars. III. Models with a single discontinuity of composition and opacity due to electron scattering. Monthly Not. Roy. Astr. Soc. **111**, 397–409 (1951).

Following the methods of the earlier papers of this series [Monthly Not. Roy. Astr. Soc. **110**, 275–286, 287–304 (1950); these Rev. **12**, 755], the authors discuss in detail the case when the coefficient of opacity is a constant independent of density and temperature and a discontinuity in molecular weight occurs somewhere near the boundary of the star. It appears that considerable extensions of the radius ("giant characteristic") can occur with a moderate fraction of the mass of the star in the envelope.

S. Chandrasekhar (Williams Bay, Wis.).

Gazaryan, Yu. L. Estimate of the gravitational action of the zodiacal light upon the motion of the perihelion of Mercury. Akad. Nauk SSSR. Astr. Žurnal **29**, 209–214 (1952). (Russian)

The author determines an upper limit to the value of Mercury's perihelion advance due to the gravitational force of the zodiacal light according to recent estimates of the density distribution of the interplanetary material. The contribution to the advance from this source is found to be so small as to be considered negligible.

R. G. Langebartel (Urbana, Ill.).

Kushwaha, R. S., and Bhatnagar, P. L. Anharmonic pulsations of Roche-model. Bull. Calcutta Math. Soc. **43**, 95–100 (1951).

The authors solve by successive approximations the adiabatic anharmonic pulsation equation for finite amplitude for the Roche model. This is intended for application to the (non-sinusoidal) velocity-time curve of the cepheid variables.

R. G. Langebartel (Urbana, Ill.).

MECHANICS

Abelé, Jean. Quelques réflexions sur la notion de point matériel. Rev. Gén. Sci. Pures Appl. **59**, 70–80 (1952).

Wunderlich, Walter. Zur Statik der Strickleiter. Math. Z. **55**, 13–22 (1951).

An isogram is a skew quadrilateral whose opposite sides are equal [G. T. Bennett, Proc. London Math. Soc. **13**, 151–173 (1914)]. The rope ladder is considered as a series of isogram linkages of constant rung length c and constant rope segment length h. When the weightless ladder, loaded at the ends, is in equilibrium, then the rope segments and the rung at a vertex lie in a plane. If the angle between planes at successive vertices along the rope is ψ (the rope twist), and the angle between planes at the ends of a rung is γ (the rung twist), Bennett showed that $\sin \psi/h = \sin \gamma/c$. It is shown that each rope takes the form of a polygon with constant rope twists, and that the rung twists are also equal. The force diagram is a double pyramid with equal dihedral angles at lateral edges, and equal dihedral angles at the base edges. The vertices of the base of this double pyramid lie on an ellipsoid of revolution whose foci are the apices of the double pyramid. These results are extended to an infinity of rungs to serve as a model for the study of the differential geometry of ruled surfaces of constant pitch. The line of striction is the locus of the midpoints of the rungs.

M. Goldberg (Washington, D. C.).

Steward, G. C. Plane kinematics. Math. Gaz. **36**, 111–120 (1952).

Expository paper.

Dobrovol'skiĭ, V. V. General theory of mechanisms for describing plane curves. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **9**, no. 36, 11–77 (1950). (Russian)

Let a range of points be represented by a fixed slotted bar m, a pencil of lines by a slotted crank revolving about a point S. If both slots are of the same width and contain a circular pin free to slide in each, a perspectivity between the range m and the pencil S is mechanized by means of this linkage. Every construction in plane projective geometry can be translated into a mechanism combined of similar linkages if some simple rules are applied. The author does this for the basic contents of this theory (generation of conic sections, etc.). *A. W. Wundheiler* (Chicago, Ill.).

Talbot, A. Equimomental systems. Math. Gaz. **36**, 95–110 (1952).

Two distributions of mass are called equimomental if, given any line in space, the two distributions have the same moment of inertia about the line. This paper gives the results of an extensive study of equimomental systems. The principal theorems are the three following. (I) Two equimomental systems have the same total mass and the same centroid. (II) If two systems of mass have the same total mass, the same centroid, and equal moments of inertia about any 6 lines (not parallel to generators of a single quadric cone), they are equimomental; and fewer lines do not suffice to insure this. (III) To any linear, planar or spatial system of positive mass there correspond infinitely many equimomental real systems of 2, 3, or 4 positive particles respectively, and none with fewer particles; with these numbers, the total available mass may be distributed among the particles in any preassigned way, and the numbers of degrees of freedom of position then remaining are 0, 1, 3 respectively. The paper includes various subsidiary theorems, and detailed discussions of various special cases.

L. A. MacColl (New York, N. Y.).

Kasner, Edward, and De Cicco, John. Geometrical properties of physical curves in space of n dimensions. Univ. Nac. Tucumán. Revista A. **8**, 127–137 (1951).

In previous papers [Proc. Nat. Acad. Sci. U.S.A. **34**, 68–72 (1948); **35**, 106–108, 201–204, 419–422 (1949); these Rev. **9**, 375; **10**, 398, 631, 747] the authors have defined the concept of physical families of curves in positional fields of force in 3-dimensional space, and have discussed properties of such families. In this paper they extend the previous studies to the case of families of curves in n-dimensional space. No motivation is given for this generalization, which appears to the reviewer to be extremely artificial.

L. A. MacColl (New York, N. Y.).

Masotti, Arnaldo. Sopra un teorema meccanico di Laisant. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **14(83)**, 388–392 (1950).

Let P_1P_2 be a path of a particle under the action of a central force, O the centre of the force, L the centre of gravity of the curve P_1P_2 on the supposition that the density at each point varies inversely as the velocity of the particle, and G the centre of gravity of the area of the sector OP_1P_2. Then the points O, G and L are collinear and $OG = \frac{2}{3}OL$ (Theorem of Laisant). In this paper it is shown that the same theorem holds for every motion in space for which the angular momentum about a fixed point O has a constant length; the angular momentum vector itself need not be constant. *J. Haantjes* (Leiden).

Singh, R. P. Poincaré's theorem and its uses. Indian J. Phys. **25**, 585–593 (1951).

The Lagrange-Jacobi virial theorem (wrongly ascribed by the author to Poincaré and Eddington) for potentials of the form $f(r^{-n})$ and $Ar^{-1}e^{-r/\lambda}$ (Yukawa potential) is found to be $\frac{1}{2}d^2I/dt^2 = 2T + n\Omega$ and $2T + \Omega - \lambda\partial\Omega/\partial\lambda$, respectively, where I is the moment of inertia about the origin, T the kinetic energy, and Ω the potential energy. These are applied for the determination of the period of radial pulsation of the Thomas-Fermi atom and of an atomic nucleus.

R. G. Langebartel (Urbana, Ill.).

Jeffreys, Harold. The case of equal periods in gyroscopic systems. J. London Math. Soc. **27**, 362–364 (1952).

An example is given which clarifies an obscure point in the theory of small oscillations about a state of steady motion. The intricacy of the question, and its comparatively

minor interest, make a more explicit description in a review unwarranted. *L. A. MacColl* (New York, N. Y.).

Haacke, Wolfhart. Die stabilen Lagen eines ebenen n-fachen Pendels mit vertikal periodisch erschüttertem Aufhängepunkt. J. Reine Angew. Math. **190**, 51–64 (1952).

An n-fold plane pendulum is a system of n simple pendulums, each of which, after the first, has its point of support fixed in the preceding pendulum. The simple pendulums move in a fixed vertical plane, and can perform complete revolutions about their points of support. This paper is concerned with an n-fold pendulum with the point of support of the first simple pendulum performing a prescribed sinusoidal motion in the vertical direction.

The system has 2^n configurations of equilibrium, in each of which some of the simple pendulums are directed up and the others down. Linearizing the equations of motion, the author studies the stability of these configurations. A linear transformation of variables with constant coefficients reduces the linearized equations of motion (corresponding to a particular configuration of equilibrium) to a system of n Mathieu equations. The stabilities of the solutions of the Mathieu equations are discussed as usual. For the configuration in question to be stable, each of the n Mathieu equations must possess two independent stable solutions. It results that the stabilities of the several configurations of equilibrium depend in an intricate way on the various parameters, including the amplitude and frequency of the motion of the first point of support. In particular, for any one of the configurations, and for fixed values of the other parameters, the frequency range is divided into certain non-overlapping intervals in which the configuration is alternately stable and unstable. By way of illustration, the case in which $n=2$ is discussed at length, with emphasis on the quantitative details. *L. A. MacColl.*

Hydrodynamics, Aerodynamics, Acoustics

Gilbarg, David. Uniqueness of axially symmetric flows with free boundaries. J. Rational Mech. Anal. **1**, 309–320 (1952).

La fonction de courant des écoulements méridiens à potentiel vérifie une équation du type elliptique dont les solutions ont la propriété de ne pas avoir d'extrêmum à l'intérieur de leur domaine de définition. Cette propriété permet d'étendre aux mouvements considérés les très simples théorèmes de comparaison qui ont été donnés par J. Serrin [dans l'oeuvre analysé ci-dessous] pour la vitesse en un point de la ligne libre dans le cas des écoulements plans. Ces théorèmes permettent alors d'établir sous des hypothèses assez larges l'unicité de la solution des problèmes du sillage infini et du jet lorsqu'il y a symétrie de révolution. *R. Gerber* (Grenoble).

Serrin, James B., Jr. Existence theorems for some hydrodynamical free boundary problems. J. Rational Mech. Anal. **1**, 1–48 (1952).

L'A. publie ses recherches consacrées à la théorie classique des sillages plans en fluide indéfini. A côté du schéma classique de Helmholtz, que nous noterons (H) (sillage indéfini limité par deux lignes libres sans point commun), l'A. envisage le schéma (F) (les deux lignes libres se rejoignent à distance finie pour ne former plus qu'une seule ligne de courant s'éloignant à l'infini) et le schéma (J), dit de jet rentrant (caractérisé, en particulier, par le recoupement de l'obstacle avec les lignes libres). Si on construit l'écoulement autour d'un obstacle donné du type (H) ou (F), on dit qu'on a résolu, pour cet obstacle, le problème déterminé (S). L'auteur se pose divers problèmes déterminés pour les obstacles doués d'une tangente bien définie en chacun de leurs points; l'angle $\psi(l)$ de cette tangente avec la direction du courant à l'infini est une fonction höldérienne de l'arc l. De plus, il existe une constante α ($\frac{1}{2}\pi \leqq \alpha \leqq \pi$) telle que $\alpha - \pi \leqq \psi(l) \leqq \alpha$. D'après celà, les catégories de profils considérés par l'A. sont assez générales pour englober le cas $\alpha = \frac{1}{2}\pi$ (étudié par Lavrentieff au moyen des procédés variationnels), et celui où $\alpha = \pi$ (traité par Leray par les méthodes des équations fonctionnelles). C'est pour la première fois, à notre connaissance, qu'on réussit à analyser par une méthode uniforme une classe aussi vaste d'écoulements.

L'objet essentiel du présent travail est l'étude des théorèmes d'existence et (sous une forme à préciser) des théorèmes d'unicité pour les problèmes déterminés (H), (S) et (J), symétriques, posés relativement aux obstacles symétriques du type ci-dessus défini. La méthode employée est celle des équations fonctionnelles de Leray-Schauder. Aussi bien, faut-il commencer par une étude qualitative à priori de la solution, supposée existante. En vue de la discussion, essentielle ici, de la forme des lignes libres, l'A. a recours à quelques propriétés extrémales des fonctions analytiques, voisines de celles qu'utilise Lavrentieff, et que l'A. se borne à énoncer, renvoyant les démonstrations à un mémoire à paraître. Du coup, on trouve des inégalités utiles pour la majoration à priori des inconnues. En fin de compte, l'A. aboutit à des énoncés d'existence et d'unité pour les problèmes (H), (S), et (J); on trouvera aussi quelques résultats relatifs au cas du détachement en proue. *J. Kravtchenko* et *R. Gerber* (Grenoble).

Serrin, James B., Jr. Uniqueness theorems for two free boundary problems. Amer. J. Math. **74**, 492–506 (1952).

Les théorèmes d'existence concernant les sillages plans qui ont fait l'objet du travail analysé ci-dessus sont complétés par une discussion d'unicité. Cette discussion repose sur deux théorèmes de comparaison, déjà utilisés par Lavrentieff, qui sont établis en toute rigueur à partir d'un théorème de Julia. Les principaux résultats sont les suivants: 1) Pour des obstacles symétriques et "réguliers" il ne peut y avoir qu'une seule solution du problème de Helmholtz; 2) pour de tels obstacles il n'existe qu'une solution du problème du sillage, celle-ci étant constituée, soit par un mouvement à la Helmholtz, soit par une cavitation à distance finie; 3) pour certains obstacles il existe un sillage à distance finie. *R. Gerber* (Grenoble).

Birkhoff, Garrett. Induced mass with free boundaries. Quart. Appl. Math. **10**, 81–86 (1952).

The author defines an induced mass tensor based on the acceleration potential and studies properties analogous to those for the usual induced mass. Since the acceleration potential, which is essentially the pressure, is constant on a free surface, the author's results can be stated for flows with free boundaries. *D. Gilbarg* (Bloomington, Ind.).

Gurevič, M. I. The impact of a plate with discontinuous jet flow. Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 116–118 (1952). (Russian)

The author considers the following problem in the two-dimensional flow of a perfect incompressible fluid. Let AB

be a fixed arc acting as an obstruction in a steady flow with free streamlines leaving the arc at either end. At time $t=0$ the arc is suddenly given a velocity v_n, a known function along the arc. Find the impulsive force on the arc. This problem is treated rather briefly with certain linearizing assumptions. Detailed computations are given in the case that AB is a straight segment and the impulsive force compared with the case when the segment rests initially on an undisturbed water surface. *J. V. Wehausen.*

Fedyaevskiĭ, K. K. Approximate theoretical determination of the added mass of rectangular plates. Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 352–354 (1952). (Russian)

Slezkin, N. A. On the impact of a plane gaseous jet on an infinite wall. Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 227–230 (1952). (Russian)

L'A. retrouve d'abord les formules relatives au choc d'une veine liquide plane contre un plan indéfini. La fonction de courant est mise sous une forme voisine de celle qu'a donnée S. Tchaplyguine pour le même problème en fluide compressible. Les résultats de Tchaplyguine sont ensuite complétés. *J. Kravtchenko* (Grenoble).

Byrd, P. F. Ergänzung zu dem Aufsatz von N. Scholz, Beiträge zur Theorie der tragenden Fläche. Ing.-Arch. **19**, 321–323 (1951).

In developing an approximate procedure for the theory of lifting surfaces based on the strip method, N. Scholz [Ing.-Arch. **18**, 84–105 (1950); these Rev. **12**, 554] has shown that four strips suffice to give exact values (according to the Birnbaum theory of lifting surfaces) for both the lift and moment of a flat plate, and that in the case of a circular arc profile four strips provide an exact value for the lift but only an approximation to the moment. The present paper extends this result to the case of n strips, and in the process gives convenient explicit formulas for the partial circulations by means of which the moment of the circular arc profile is shown to approach the exact value as $n\to\infty$. *D. Gilbarg* (Bloomington, Ind.).

Müller, W. Bewegung des langgestreckten Rotationskörpers in einer zur Längsachse geneigten Richtung. Ing.-Arch. **20**, 57–66 (1952).

In an earlier paper [Ing.-Arch. **19**, 282–295 (1951); these Rev. **13**, 503] the author considered bodies of revolution determined by source-sink distributions $f(x)$ along a segment $[-1, 1]$ of the x-axis where this was also the direction of flow. In the present paper he takes the direction of flow along the y-axis and considers the bodies of revolution obtained by a distribution of dipoles $F(x)$ along the same segment $[-1, 1]$, the axes of the dipoles in the y-direction [cf. the reference cited in the review of the earlier paper]. In order to treat a flow making an angle with the axis of symmetry one must be able to relate $f(x)$ and $F(x)$ for the same body. The author discusses an approximate solution of this problem, starting with the relation $f(x)=-kF'(x)$ which is exact for an ellipsoid ($k=1$). *J. V. Wehausen.*

Mohr, Ernst. Der Beschleunigungswiderstand bewegter Körper in einer Flüssigkeit. Z. Angew. Math. Mech. **32**, 87–88 (1952).

Let W' be the pressure resistance when a body moves without rotation in the direction of the x-axis, with acceleration du/dt, in incompressible inviscid fluid, otherwise at rest. Let W be the pressure resistance when the body is regarded as at rest and the fluid accelerated. S. Neumark [same Z. **16**, 117–120 (1936)] showed that $W-W'=Mdu/dt$, where M is the mass of fluid displaced by the body. The author remarks that this result follows without calculation since the pressure field in the second case can be regarded as that in the first increased by $\rho x du/dt$ (the author omits the x). He further observes that the argument is unchanged if the fluid is viscous. Moreover the formula can be extended to any motion such that axes fixed in the body remain fixed in direction in space. *L. M. Milne-Thomson.*

Dolidze, D. E. The unsteady motion of a viscous fluid about a cylinder. Soobščeniya Akad. Nauk Gruzin. SSR. **8**, 11–18 (1947). (Russian)

Dolidze, D. E. Unsteady motion of a viscous fluid about a sphere. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **16**, 105–116 (1948). (Russian. Georgian summary)

The author considers the limiting behavior of the solutions for two problems of unsteady flow, the first paper treating the 2-dimensional and the second the 3-dimensional cases. The Navier-Stokes equations are linearized by neglecting inertial terms but the author indicates that this is not essential. In order to insure uniqueness in each case it is assumed that certain continuity requirements are satisfied and that $\lim_{r\to\infty} r^2 p v_i = \lim_{r\to\infty} r^3 v_i\, \partial v_i/\partial x_k = 0$, where p is pressure, v_i velocity in direction x_i and $r^2=\sum x_i^2$ [cf. Dolidze, Akad. Nauk SSSR. Prikl. Mat. Meh. **11**, 237–250 (1947); **12**, 165–180 (1948); these Rev. **9**, 116, 630]. The first problem is that of a circular cylinder (sphere) rotating with velocity $\Omega(t)$. Here it is shown that with $\Omega(0)=0$, $\lim_{t\to\infty}\Omega(t)=\Omega_0$, the explicitly constructed solution to the unsteady problem converges to the solution of the steady problem ($\Omega=\Omega_0$). In the second problem the cylinder (sphere) is taken to move rectilinearly with velocity $v_0(t)$. An explicit solution is constructed with $v_0(0)=0$, but it is shown that a limiting solution does not exist for $\lim_{t\to\infty} v_0(t)=v_0$. Since in this case no steady state solution exists (in the necessary sense), the two problems illustrate a theorem of the author stating that for the limit of the solutions of the unsteady problem to exist the corresponding steady state problem must have a solution. *J. V. Wehausen.*

Hess, Robert V. A solution of the Navier-Stokes equations for source and sink flows of a viscous heat-conducting compressible fluid. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2630**, 60 pp. (3 plates) (1952).

Les effets de la viscosité sur l'écoulement d'un fluide compressible et conducteur sont bien connus lorsqu'il s'agit de mouvements à une dimension [voir en particulier Lagerstrom, Cole and Trilling, Problems in the theory of viscous compressible fluids, Calif. Inst. Tech., Pasadena, 1949; ces Rev. **12**, 873]. Le but de cette note est d'étudier ces effets de la viscosité pour les mouvements à deux et trois dimensions. Le schéma considéré est celui d'un puits ou d'une source; une solution des équations de Navier-Stokes est donnée dans l'hypothèse des écoulements à énergie totale constante. Ceci suppose une valeur particulière du nombre de Prandtl et du rapport des deux coefficients de viscosité. Une discussion de la valeur de ce dernier rapport est faite en utilisant la théorie cinétique des gaz. Par ailleurs le cas de l'énergie totale variable est examiné.

L'étude théorique précédente permet de mettre en relief les principales caractéristiques de l'effet longitudinal, l'effet transversal étant nul, pour les mouvements plans et dans

l'espace. L'ordre d'importance de cet effet est exprimé à l'aide d'un paramètre sans dimensions, inverse d'un nombre de Reynolds. *R. Gerber* (Grenoble).

Šapošnikov, I. G. On the computation of diffusion phenomena in the equations of hydrodynamics. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **21**, 1309–1310 (1951). (Russian)

Sleskin [Doklady Akad. Nauk SSSR **77**, 205–208 (1951); ces Rev. **13**, 81] et Vallander [ibid. **78**, 25–27 (1951); ces Rev. **13**, 81] ont proposé une nouvelle forme du vecteur transport de masse au sein d'un gaz (noté classiquement ρV) afin de tenir compte des phénomènes de la diffusion moléculaire et thermique. L'A. présente ici une critique des travaux ci-dessus et conclut que les hypothèses de Sleskin et Vallander ne correspondent pas à la réalité physique.

J. Kravtchenko (Grenoble).

Schröder, Kurt. Verwendung der Differenzenrechnung zur Berechnung der laminaren Grenzschicht. Math. Nachr. **4**, 439–467 (1951).

The laminar boundary layer equation is solved by a combined iteration and finite difference method. The convergence of the finite difference part is referred to known theorems. The proof of convergence for the iteration process is carried out under certain conditions which can be checked in a practical calculation. Short cuts for practical calculation are given; several examples are worked out. In the case of the Schubauer ellipse, the point of separation is correctly predicted by an extrapolation of the skin friction.

C. C. Lin (Cambridge, Mass.).

Schröder, K. A simple numerical method for the calculation of the laminar boundary layer. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1317**, 47 pp. (1952).

Translated from Zentrale für wissenschaftliches Berichtswesen der Luftfahrtforschung des Generalluftzeugmeisters (ZWB), Forschungsbericht Nr. **1741** (1943).

Timman, R. La méthode des caractéristiques et la calculation de la couche limite laminaire en écoulement tridimensionel. Nationaal Luchtvaartlaboratorium, Amsterdam. Report F. **62**, i+7 pp. (1950).

The momentum equations are derived for the three-dimensional boundary layer in a manner similar to that in the Kármán-Pohlhausen method. For the representation of the velocity profile, a two-parameter family of curves is used. The resultant equations are two quasi-linear partial differential equations of the first order which the author suggests integrating by the method of characteristics. *C. C. Lin.*

Millsaps, Knox, and Pohlhausen, Karl. Heat transfer by laminar flow from a rotating plate. J. Aeronaut. Sci. **19**, 120–126 (1952).

The complete solution of the laminar boundary layer on a rotating flat plate previously given by von Kármán [Z. Angew. Math. Mech. **1**, 233–252 (1921), p. 245] and Cochran [Proc. Cambridge Philos. Soc. **30**, 365–375 (1934)] for the velocity distribution is extended to give the temperature distributions including the effects of viscous dissipation. The general equations are reduced to a set of simultaneous equations which are solved numerically. There are obtained three generalized functions determining the temperatures, each dependent on the Prandtl number and a dimensionless distance above the plate. These, as well as similar functions for the velocity distribution may be applied to any particular fluid and plate rpm. *N. A. Hall.*

Rotta, J. Statistische Theorie nichthomogener Turbulenz. II. Z. Physik **131**, 51–77 (1951).

The author applies his theory of part I [same Z. **129**, 547–572 (1951); these Rev. **13**, 596] to the case of two-dimensional channel flow, after introducing certain assumptions. The distributions of mean velocity and turbulence level are found to be in reasonable agreement with experiments.

C. C. Lin (Cambridge, Mass.).

Batchelor, G. K. Diffusion in a field of homogeneous turbulence. II. The relative motion of particles. Proc. Cambridge Philos. Soc. **48**, 345–362 (1952).

The author continues his development of the theory of diffusion resulting from turbulent motion. The results of part I [Australian J. Sci. Research. Ser. A. **2**, 437–450 (1949); these Rev. **12**, 451] are briefly reviewed and extended somewhat. In that part the author had treated the diffusion of a single particle relative to a fixed origin. He now wishes to discuss the statistical properties of a pair of particles in a homogeneous turbulence. For this he considers the probability density function $Q(\mathbf{X}, \mathbf{Y}, t|\mathbf{Y}_0, t_0)$ that two particles P, Q, with $\mathbf{Y}_0$ the vector PQ at time t_0, will have P displaced by the vector $\mathbf{X}$ and have PQ be the vector $\mathbf{Y}$ at time t. The behavior of this function and related ones is examined in various limiting situations and under various assumptions which cannot be outlined briefly. The possibility that $Q(\mathbf{Y}, t|\mathbf{Y}_0, t_0)$ may satisfy a differential equation is discussed and one is proposed for a certain region of the variables. *J. V. Wehausen* (Providence, R. I.).

Corrsin, Stanley. On the spectrum of isotropic temperature fluctuations in an isotropic turbulence. J. Appl. Phys. **22**, 469–473 (1951).

The author carries out the analysis indicated in the title to obtain results analogous to those for velocity fluctuations in isotropic turbulence. In particular, it is found that the spectrum begins with the second power of the wave number k at the lower end, in contrast to the fourth-power behaviour for velocity spectrum. The other power behaviours ($k^{-5/3}$ for the intermediate range and k^{-7} for the high frequency range) are found to be the same as in the velocity case, following similar arguments. The existence of the $k^{-5/3}$ range for temperature depends on that for velocity.

C. C. Lin (Cambridge, Mass.).

Lighthill, M. J. On sound generated aerodynamically. I. General theory. Proc. Roy. Soc. London. Ser. A. **211**, 564–587 (1952).

The author considers the following problem. Suppose that in a very large volume of fluid V the fluid in one part P is moving in a turbulent or fluctuating manner; the rest of the fluid is at rest except for the motion associated with sound waves generated by the fluctuating motion in P. The author wishes to estimate this sound. After a discussion of physical ideas he starts with the equations

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_i}(\rho v_i)=0, \quad \frac{\partial}{\partial t}(\rho v_i)+a_0^2\frac{\partial \rho}{\partial x_i}=-\frac{\partial T_{ij}}{\partial x_j},$$

where a_0 is the sound velocity and $T_{ij}=\rho v_i v_j+p_{ij}-a_0^2\rho\delta_{ij}$. In P the tensor T_{ij} may be approximated by $\rho v_i v_j$ and in $V-P$ by 0. The fluctuations of density may then be de-

duced as

$$\rho-\rho_0=\frac{1}{4\pi a_0^2}\frac{\partial^2}{\partial x_i\partial x_j}\int T_{ij}\left(\mathbf{y},\, t-\frac{|\mathbf{x}-\mathbf{y}|}{a_0}\right)\frac{dy}{|\mathbf{x}-\mathbf{y}|},$$

which may be interpreted as that caused by a distribution of quadrupoles. Associated formulas for the intensity field and acoustic power output are given. The similarity theory for such sound production is discussed. Finally, the necessary modifications are introduced for the case when the region P is in uniform motion. *J. V. Wehausen.*

Hasimoto, Hidenori. On the asymptotic behaviour of three-dimensional compressible fluid flow at a great distance from a body. I. The force and moment on a solid body in a stream of compressible fluid. J. Phys. Soc. Japan 6, 383–387 (1951).

The author expands the velocity potential function for steady incompressible flow into a series

$$\Phi=U[r\cos\theta+A_1r^{-1}+A_2r^{-2}+\sum_3^\infty A_n(\theta,\omega)r^{-n}],$$

where r, θ, ω are spherical polar coordinates, $(U, 0, 0)$ is the cartesian velocity at infinity, corresponding to Mach number M, $A_1=a_0(1-M^2\sin^2\theta)^{-\frac{1}{2}}$,

$$A_2=(1-M^2\sin^2\theta)^{-1.5}\times[a_{1,0}\cos\theta+\sin\theta(a_{1,1}\cos\omega+a_{1,-1}\sin\omega)],$$

$a_0, \cdots, a_{1,-1}$ are constants, and the exact forms of A_n are not required here. By considering the balance of momentum and angular momentum of the fluid in a large sphere $r=$const. and passing to the limit $r\to\infty$ he obtains for the force on the body the cartesian components $-\rho_\infty Q_\infty U, 0, 0$, and for the moment 0,

$$4\pi a_{1,-1}\rho_\infty U^2/(1-M^2),\quad -4\pi a_{1,1}\rho_\infty U^2/(1-M^2).$$

Here ρ_∞ is the density at infinity, and $\rho_\infty Q_\infty$ is the total outward rate of mass flow at infinity, originating from sources in the body. *J. H. Giese* (Havre de Grace, Md.).

Rao, G. V. R. Two-dimensional subsonic flow past elliptic cylinder by the variational method. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 2666, 39 pp. (1952).

The variational method of C. T. Wang is used to calculate subsonic compressible flow past elliptic cylinders at zero incidence. An integral J which is stationary for the correct solution was given by Bateman following Hargreaves. Wang's method is an improvement on earlier applications of the variational principle in which conditions at infinity had been disregarded.

In the present paper the integral J is expressed in terms of elliptic coordinates, and the difference between the compressible and incompressible solutions is determined approximately by the method of Rayleigh-Ritz. Thus, this difference is represented as a linear combination of nine "modal functions" whose coefficients are determined so as to make J stationary. The modal functions chosen are reasonable but, inevitably, somewhat arbitrary.

Comparison with other approximate results shows general agreement, but the present results tend to deviate to a greater extent from the incompressible solutions. It would appear difficult theoretically to give preference to the results of one or the other of these methods owing to the nonexistence of numerical bounds which would indicate the quality of the various approximations. *A. Robinson.*

Germain, P. Sur l'approximation homographique dans l'étude des fluides compressibles. Recherche Aéronautique no. 25, 9–17 (1952).

To investigate transonic flows of a two-dimensional compressible fluid the author considers the equation $E(k)=k(z)u_{xx}+u_{zz}=0$ of mixed type, where the subscripts as usual represent partial differentiations; x and z are the independent real variables. As a first approximation set $k(z)=Cz$, C being a convenient constant, to obtain the well-known Tricomi equation. In this paper the author investigates the equation $E(z/(z+a))=0$, i.e., when $k(z)=z/(z+a)$, $a>0$, maintaining that this homographic approximation is a better representation of transonic phenomena. By linear transformations of dilation of the independent variables the Tricomi equation can then be shown to be a limiting case of $E(z/(z+a))=0$. From a simple solution the author builds up new solutions by means of an integral operator which is essentially the convolution. By means of this integral operator solutions with singularities are constructed and finally he obtains an integral representation for a fundamental solution. *A. Gelbart* (Trondheim).

Couchet, G. Sur une généralisation de la transformation de Joukowsky. Recherche Aéronautique no. 20, 23 (1951).

Timman, R., van de Vooren, A. I., and Greidanus, J. H. Aerodynamic coefficients of an oscillating airfoil in two-dimensional subsonic flow. J. Aeronaut. Sci. 18, 797–802, 834 (1951).

The title problem is attacked by introducing elliptic coordinates and developing the acceleration potential in Mathieu functions (cf. bibliography of paper for previous work). The results exhibit discrepancies with those of Schade and Dietze, particularly at the higher frequencies. Moreover, the authors' results have been criticized by Fettis [same J. 19, 353–354 (1952)] as being mutually inconsistent. [Reviewer notes: (a) Errors at high frequencies could be due simply to poor convergence of the Mathieu function expansions; cf. Morse and Rubenstein, Physical Rev. 54, 895–898 (1938), where an analogous diffraction problem is treated. (b) If, as the authors state, their "regular" solution is finite at the leading edge, the superposition of their "singular" solution, which must be singular at the trailing (as well as the leading) edge would violate the (Kutta) condition of continuity of pressure at the trailing edge. However, it is probable that the "regular" solution does exhibit singularities at both edges, in the neighborhood of which the solutions may be non-uniformly convergent.]

J. W. Miles (Los Angeles, Calif.).

Nelson, Herbert C., and Berman, Julian H. Calculations on the forces and moments for an oscillating wing-aileron combination in two-dimensional potential flow at sonic speed. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 2590, 36 pp. (1952).

Linearized theory (valid for two-dimensional sonic flight at sufficiently high values of the reduced frequency parameter) is used to solve the problem of the title. (Previous work of Rott [Z. Angew. Math. Physik 1, 380–410 (1950); these Rev. 12, 874] did not include the aileron.) The results are presented in extensive tables and applied to bending-torsion flutter of a wing sans aileron. Several shortcomings of the results are discussed, but the authors do not consider the possible importance of edge (i.e., three-dimensional) effects. In the opinion of the reviewer, the last will be decisive at

the sonic speed and a flutter analysis based on two-dimensional aerodynamic theory would yield completely misleading results. *J. W. Miles* (Los Angeles, Calif.).

Rott, N. On the unsteady motion of a thin rectangular wing in supersonic flow. J. Aeronaut. Sci. 18, 775–776 (1951).

Analogy with the half-plane diffraction problem is used to obtain the solution for arbitrary time dependence but no spanwise variation of prescribed wing motion, following Lamb's analysis [Proc. London Math. Soc. (2) 4, 190–203 (1906); 8, 422–437 (1910)]. The potential on the wing and inside the tip Mach line is related explicitly to the two-dimensional potential outside the tip Mach cone. The spanwise integral of the potential ϕ on the wing from the edge to the Mach line is shown to be the same integral of ϕ less half the vertical integral of ϕ from the wing to the Mach cone. The results agree with those previously known for special cases.

Reviewer's notes. (a) The author's comments on the derivation of Eq. (6) presumably imply introduction of parabolic coordinates and invocation of appropriate boundary conditions on the Mach cone. (b) Radical in Eq. (11) should contain $(z-v)$. (c) Author's remark notwithstanding, higher-order solution of form $r^{-n-\frac{1}{2}}\cos(n+\frac{1}{2})\theta F_n(x, t, r)$ represents $(\partial/\partial y)^{n+1}\phi_n$, where ϕ_n exhibits the required behavior at the wing edge ($y=0$) and the prescribed wing motion of spanwise variation y^n. The reviewer has obtained the general solution in this manner but believes an alternative solution (unpublished) to be simpler. *J. W. Miles.*

Hayes, W. D., Roberts, R. C., and Haaser, N. Generalized linearized conical flow. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 2667, 48 pp. (1952).

This is a study of small-perturbation, supersonic, potential flows for which the velocity potential ϕ is homogeneous of degree n in a Cartesian system whose origin is placed at the nose of a thin wing. After a review of conical-flow theory ($n=1$), the authors adopt a suggestion attributed to G. N. Ward; viz., the use of the variable ξ, as follows: Let $\tau(x, y, z)$ be an elementary homogeneous particular solution of the wave equation of degree 0; then

$$\xi=x_0\alpha(\tau)+y_0\beta(\tau)+z_0\gamma(\tau)=\xi(x_0, y_0, z_0; x, y, z)$$

and $\xi(x_0, y_0, z_0; x_0, y_0, z_0)=0$. It is shown that the velocity potential can be expressed in the form $\phi(x_0, y_0, z_0)=\int^{\tau_0} g(\xi, \tau)d\tau$. The choice $g(\xi, \tau)=\xi f'(\tau)$ leads to familiar conical-flow solutions, upon proper choices of τ for regions inside and outside the Mach cone. For "second-degree conical" flow $g(\xi, \tau)$ is taken equal to $\frac{1}{2}\xi^2 f'(\tau)$. Examples worked out include rolling and pitching wings and a family of swept delta wings (quadrilateral planforms). The extension to unsteady flows is carried out for quasi-steady motion and for harmonic motion. *W. R. Sears* (Ithaca, N. Y.).

Holt, M. The flow of two adjacent plane supersonic jets past flat-plate wings. II. Quart. J. Mech. Appl. Math. 4, 419–431 (1951).

"The method of determining the pressure distribution on delta wings in a non-uniform supersonic stream developed" in the author's previous paper [same J. 3, 200–216 (1950)] "is extended to include wings on which subsonic conditions apply along part or the whole of the leading edge." (From the author's summary.) *J. W. Miles.*

Goldsworthy, F. A. Supersonic flow over thin symmetrical wings with given surface pressure distribution. Aeronaut. Quart. 3, 263–279 (1952).

The author's summary reads in part: "The inverse problem of determining the supersonic flow past a thin symmetrical wing at zero incidence with given surface pressure distribution is solved for wings of arbitrary plan form. Expressions are obtained for the vertical derivative of the potential on the wing surface from which, using the linearised boundary condition of zero normal velocity at the surface, the profile of the wing can be designed. The integral equation method adopted by Evvard [Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1382 (1947); these Rev. 8, 610] and extended by Ward [Quart. J. Mech. Appl. Math. 2, 136–152 (1949); these Rev. 11, 64] is used. The analysis cannot be applied to pointed wings, whose leading edges are subsonic." The results are applied to rectangular and arrowhead wings (with supersonic leading edges). The reviewer notes that Eq. (6) is completely missing, Eq. (5) having been reprinted instead. *J. W. Miles* (Los Angeles, Calif.).

Drougge, Georg. Wing sections with minimum drag at supersonic speeds. Flygtekn. Försöksanstalt. Rep. no. 26, 15 pp. (1949).

A treatment of the familiar minimum-wave-drag problems of linear two-dimensional (Ackeret) airfoil theory: If $y(x)$ is the (symmetrical) profile shape, $\int_{-1}^{1} y'^2 dx$ is minimized for constant $\int_{-1}^{1} y^n dx$, with $n=1$, 2, and 3, and also for constant maximum y. The profiles are determined and plotted, and the drag is calculated for each case. Then, considering the effect of skin-friction drag, the optimum thickness ratio for each case is determined. *W. R. Sears* (Ithaca, N. Y.).

Morikawa, George K. A non-planar boundary problem for the wave equation. Quart. Appl. Math. 10, 129–140 (1952).

The author considers linearised supersonic flow past a half-plane (or rectangular wing of large aspect ratio) at incidence α, which is attached to a circular body at zero incidence, infinite (or sufficiently long) in both directions. This problem is rightly regarded as fundamental for the theory of wing-body interference. The Laplace transformation is applied to the coordinate in the free-stream direction and the transformed problem is solved by a Green's function method. It is found that the transformation back to the physical plane presents considerable analytical difficulties. For that reason the author studies first two simplified problems: (i) the body is replaced by a flat plate approximately in the plane of the wing but at zero incidence; (ii) the body has infinite radius, i.e., it is an infinite plate normal to the wing. An approximate Green's function is then suggested for the more general case and this is transformed back to the physical plane, again approximately. It appears that the approximations are satisfactory at least near the wing-body junction at the leading edge and far down-stream. Quantitative results are shown on graphs. To appreciate the value of the author's contribution it should be remembered that exact solutions for such problems are exceedingly difficult to obtain.

It is not clear to the reviewer why the complementary case—wing at zero incidence, body at incidence α—is dismissed in a few sentences. However it may be that this case is considered more fully in two longer papers to which the author refers [a thesis and a California Institute of Technology report, both unpublished]. *A. Robinson.*

Kirkby, S., and Robinson, A. Interference on a wing due to a body at supersonic speeds. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2500** (10,631), 10 pp. (1952).

This is an early (1947) calculation by the simple strip method on the increment of lift on a rectangular wing by the induced velocity of a cone body at a small angle of attack in supersonic flight. In the appendix, the authors show that when the wing chord is very small in comparison with the span, the results of the strip method agree with those of the exact linear theory. *H. Tsien.*

Goldsworthy, F. A. Two-dimensional rotational flow at high Mach number past thin aerofoils. Quart. J. Mech. Appl. Math. **5**, 54–63 (1952).

By a careful analysis of the relative orders of magnitudes in hypersonic inviscid flow over a two-dimensional thin airfoil, the author definitely shows that if quantities of the order M_0^{-2} (M_0 free stream Mach number) are neglected, the hypersonic similarity rule should hold even with variable entropy, and that the equations of motion then reduce to those for one-dimensional unsteady flow. Hayes [Quart. Appl. Math. **5**, 105–106 (1947); these Rev. **8**, 610] obtained these results by intuitive argument. Then by using a result of Lighthill [Philos. Mag. (7) **41**, 1101–1128 (1950); these Rev. **13**, 400], the author shows that the effects of entropy change are, in fact, quite small, and therefore the previous calculations of Linnell using the shock-expansion method [J. Aeronaut. Sci. **16**, 22–30 (1949); these Rev. **10**, 641] are indeed valid. *H. Tsien* (Pasadena, Calif.).

Eggers, A. J., Jr., and Syvertson, Clarence A. Inviscid flow about airfoils at high supersonic speeds. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2646**, 65 pp. (1952).

Assuming the fluid to be a diatomic gas satisfying the equation of states for the perfect gas, the authors first develop the appropriate equations for the analysis of two-dimensional inviscid flows with variable specific heats. The authors then derive the condition for the validity of shock-expansion method as

$$\text{(A)} \qquad \left|\frac{\partial\delta/\partial C_1}{\partial\delta/\partial C_2}\right| \ll 1$$

where δ is the flow direction, C_1, C_2 are characteristic parameters; C_1 constant along the wave reflected from the nose-shock, C_2 constant along waves generated by the curvature of the airfoil surface. By taking the value of this "disturbance strength ratio" immediately downstream of the nose as representative, the authors then show that for ratios of specific heats $\gamma > 1.3$, and free stream Mach number M_0 greater than 3, the condition is satisfied for nose flow deflection angle δ_N less than 40°. Therefore for this wide range of parameters, the shock-expansion method gives accurate results. This is in agreement with the conclusion of Goldsworthy [see the preceding review]. For thin airfoils in hypersonic flows, the author gives a simplified shock-expansion method which satisfies the hypersonic similarity law, and uses an average γ_a. Here γ_a is a function of free stream temperature T_0 and $M_0\delta_N$ and is tabulated for $T_0 = 500°R$. As $\gamma \to 1$, (A) is no longer satisfied, the shock-expansion method then fails. But this value of γ is connected with very large M_0, then Busemann's method based upon $M_0 \to \infty$, $\gamma = 1$ could be used with advantage. *H. Tsien.*

Eggers, A. J., Jr., and Savin, Raymond C. Approximate methods for calculating the flow about nonlifting bodies of revolution at high supersonic airspeeds. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2579**, 40 pp. (1951).

As J. W. Maccoll has indicated sometime ago [Proc. Roy. Soc. London. Ser. A. **159**, 459–472 (1937)], the solution of flow over an unyawed cone at high supersonic speed can be considerably simplified because of the proximity of the shock and the cone surface. This fact together with the modern concept of hypersonic similarity has been utilized recently by L. Lees [J. Aeronaut. Sci. **18**, 700–702 (1951)] in giving a very simple solution. In the first part of the present report, the authors obtain other approximations of conical flow at hypersonic speeds for cones of both small and large vertex angles. The analytical expressions are, however, rather complex in comparison with those of Lees. In the second part, the authors show that for slender bodies in hypersonic flow, the expansion flow along the ogive after the vertex can be approximated by the plane flow, neglecting the additional term from the axial symmetry. Plane expansion after the nose shock can, however, be approximated by the simple Prandtl-Meyer flow, as shown recently by F. A. Goldsworthy [see the second preceding review]. Therefore the complete solution is quite simple. Numerical calculations show that the agreement of the proposed approximate solution and the "exact" solution with numerical integration along the characteristics is within the accuracy of the characteristic method. *H. Tsien* (Pasadena, Calif.).

Polachek, H., and Seeger, R. J. On shock-wave phenomena; refraction of shock waves at a gaseous interface. Physical Rev. (2) **84**, 922–929 (1951).

The relations governing the refraction of a shock wave at a gaseous interface contain five parameters, a fixed set of parameters determining solutions that are not necessarily unique. Among the solutions two types of refraction patterns appear in general, one a three-wave configuration at the interface with a reflected shock, the other with a reflected rarefaction wave. The authors select as the physically plausible solutions those which connect continuously with the known solutions for the limiting case of an acoustic wave and for the case of a finite shock at normal incidence. They calculate numerically families of solutions for a number of gas combinations at varying angles of incidence of the shock wave and study in this way the dependence of the different configurations on the parameters involved. Two features of the solutions obtained are: (1) regular refraction (three-wave configuration at the interface) does not occur at glancing incidence; and (2) in the region of regular refraction there is no total reflection of finite shock waves (unlike the case of acoustic waves). The paper contains a photograph supporting the authors' calculated results.

D. Gilbarg (Bloomington, Ind.).

Herpin, A. Théorie générale des ondes de choc. Mémorial de l'Artillerie Française **24**, 851–897 (1950).

Expository paper.

Ferrari, Carlo. Velocity and temperature distribution through the laminar boundary layer in supersonic flow. J. Aeronaut. Sci. **19**, 39–47, 65 (1952).

An approximate procedure is developed which may be applied to a compressible gas with arbitrary pressure distribution along the boundary in the external stream. The usual boundary-layer equations are applied including the assumptions of constant static pressure. The equations are

modified by the von Mises transformation so that the dimensionless distance from the wall and the stream functions become independent variables. The resulting energy and momentum equations are then solved approximately both in the neighborhood of the free stream and the wall taking account of the respective boundary conditions. These external and internal solutions are made to fit at an intermediate point determined by the inflection in the external solution for dimensionless kinetic energy vs. reduced stream function. Although the analysis appears involved the method is well suited to numerical computions as evidenced by a well selected example concluding the paper. *N. A. Hall.*

Van Driest, E. R. Turbulent boundary layer in compressible fluids. J. Aeronaut. Sci. **18**, 145–160, 216 (1951).

The author applies the momentum transfer theory to the boundary layer in a compressible fluid and computes the velocity distribution to the approximation that a logarithmic distribution is obtained in the incompressible case. Temperature distribution is also computed. By assuming the constant in the theory to be independent of Mach number, the author obtains the coefficients of skin friction and heat transfer for various Mach numbers. It is found that the skin friction coefficient decreases with Mach number, and compares roughly with the available experimental data.

C. C. Lin (Cambridge, Mass.).

Roy, Maurice. Complément à l'analyse de la structure des quasi-ondes de choc et combustion. C. R. Acad. Sci. Paris **234**, 268–271 (1952).

For the combustion wave where $D \cong \theta \cong X$, the author assumes a simple law for the heat release for $\xi > 0$. The rate of heat release is assumed to jump from 0 at $\xi = 0^-$ to a finite value at $\xi = 0^+$, rises parabolically to a maximum at $\theta = \frac{1}{2}\theta_2$, and then drop to zero again at $\theta = \theta_2$ ($\alpha = \alpha_2$) (subscript 2 denotes quantities at end of the combustion wave). Therefore the complex problem of chemical kinetics is not considered. The treatment is thus somewhat similar to that of R. Friedman and E. Burke [J. Aeronaut. Sci. **18**, 239–246 (1951); these Rev. **12**, 649]. For the reaction zone $\xi > 0$, let $a = (\theta_0 - 1)/(\theta_2 - \theta_0)$, $v = \exp(-a\xi)$. Up to first-order terms in ϵ, the author gives

$$\theta \cong \theta_2 - (\theta_2 - \theta_0)v + \frac{\gamma - 1}{2\gamma}\epsilon(\theta_2 - \theta_0)^2 v(1 - v),$$

$$\chi \cong \theta_2 - (\theta_2 - \theta_0)v + \epsilon\left\{\frac{\gamma - 1}{2\gamma}[\theta_2^2 - (\theta_2^2 - \theta_0^2)v] + [\theta_2 - (\theta_2 - \theta_0)v] \times \left[\frac{\gamma + 1}{2\gamma}\theta_2(1 - v) - \frac{\gamma - 1}{2\gamma}\theta_0 v - (1 - v)\right]\right\}.$$

By joining the solutions for the pre-reaction zone with that for the reaction zone, at $\xi = 0$, the plane speed u_1, is determined. In terms of the maximum A_M of the reaction rate Da/Dt at $\theta = \frac{1}{2}\theta_2$, it is

$$u_1 = 4\frac{\theta_2 - \theta_0}{\theta_2}\sqrt{\frac{\eta_1 A_M}{3\rho_1(\theta_0 - 1)}}.$$

H. Tsien (Pasadena, Calif.).

Bultot, F. Sur la courbure des surfaces de discontinuité de l'atmosphère. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **37**, 977–985, 1081–1092 (1951).

The curvature of a surface of discontinuity is calculated and expressed in terms of the space-derivatives of the first and second order of the pressure, which must itself be continuous across such a surface in the atmosphere. If the surface is cylindrical with generators horizontal, an approximate expression for the curvature is found in the geostrophic case and applied to the case of a warm air-mass overlying a cold one. The upward or downward curvature of the surface of discontinuity then depends, in the main, on the discontinuity in the lapse-rates in the warm and cold air. A surface of discontinuity across which the temperature is continuous, but its derivatives are not, is also discussed and applied to the case of the tropopause.

G. C. McVittie (Urbana, Ill.).

Faran, James J., Jr. Sound scattering by solid cylinders and spheres. J. Acoust. Soc. Amer. **23**, 405–418 (1951).

The scattering of sound by solid infinite circular cylinders and by solid spheres which can support shear as well as compressional waves is considered. The solution is given in terms of the surface value of the logarithmic derivative of each mode of the external pressure expanded either in cylindrical or spherical coordinates as the case may be. These derivatives are determined by applying the appropriate joining conditions at the surface of the scattering which include (1) continuity of normal displacement, (2) continuity of normal pressure, and (3) zero tangential stress. The results are evaluated for a number of cases and tested experimentally. Resonances occur near the frequencies of free vibration for the scatterer. *H. Feshbach.*

Ghosh, R. N. Application of perturbation method to acoustical problems. Bull. Allahabad Univ. Math. Assoc. **15**, 17–23 (1951).

A review of the perturbation methods employed in the discussion of (1) coupled rectangular rooms with rigid walls, (2) a rectangular room whose walls are not hard, i.e., the velocity potential satisfies the boundary condition $\partial\psi/\partial n = F(S)\psi$. No references are given. *H. Feshbach.*

Elasticity, Plasticity

Galimov, K. Z. An invariant form of the compatibility conditions of finite deformations. Doklady Akad. Nauk SSSR (N.S.) **77**, 577–580 (1951). (Russian)

Invariant forms of the compatibility conditions for bodies undergoing finite deformations are found both in Lagrangian and Eulerian systems of coordinates. *H. I. Ansoff.*

Novožilov, V. V. On the relation between stress and strain in a nonlinear elastic medium. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 183–194 (1951). (Russian)

In his textbook on nonlinear theory of elasticity [Foundations of nonlinear theory of elasticity, Moscow-Leningrad, 1948; these Rev. **12**, 651] the author developed generalized stress-strain equations in terms of the derivatives of the strain energy function. In this paper he interprets these relations in terms of three generalized moduli of nonlinear elasticity: a generalized bulk modulus, a generalized shear modulus and a third one which is defined as the "phase angle between the stress and strain deviators." The stress-strain relations are derived in terms of these moduli and some implications with respect to the theory of plasticity and anisotropic behavior are pointed out. *H. I. Ansoff.*

Green, A. E., Rivlin, R. S., and Shield, R. T. General theory of small elastic deformations superposed on finite elastic deformations. Proc. Roy. Soc. London Ser. A. 211, 128–154 (1952).

Cauchy [Oeuvres, sér. 2, t. 9, Gauthier-Villars, Paris, 1891, pp. 162–173, 342–369, see eqns. (36) and (37) in each] in 1829 obtained a formula for the relation between stress and strain in a perfectly elastic anisotropic body suffering a displacement with small gradients from a state of given stress and displacement from its natural state. [Pearson's criticism, cited by Love, Mathematical theory of elasticity, 4th ed., Cambridge, 1927, §75, of St. Venant's derivation is correct only because of the latter's faulty formulation of the general theory; Cauchy's result has been derived correctly by Murnaghan [Anais Acad. Brasil. Ci. 21, 329–336 (1949); these Rev. 11, 627].] Among various corollaries of this basic formula is the fact that the apparent elastic behavior of an initially stressed isotropic body is no longer isotropic.

The authors study the same problem under more restrictive hypotheses, which enable them to get a simpler and more usable result. First, they write the total displacement v' from the natural state in the form $v'=v+\epsilon w$, where v is the known displacement corresponding to the known initial stress, and ϵ is a parameter whose powers higher than 1 are to be neglected systematically. Thus they suppose that not only the gradients but also the second displacement itself is very small. Second, they restrict attention to isotropic bodies. For the additional stress required to produce the small deformation they then obtain formulae involving nine coefficients, themselves determinable from the known finite deformation.

They next work out the specific formulae for the case when the large initial deformation is homogeneous. They remark that the solution for small twist of a finitely extended cylinder previously obtained by Green and Shield [Philos. Trans. Roy. Soc. London. Ser. A. 244, 47–86 (1951); these Rev. 13, 509] might have been obtained by specializing these results. For the case when two of the principal extensions are equal, they reduce the general problem to that of finding two appropriate harmonic functions. They outline an application to the problem of a semi-infinite incompressible stressed solid depressed by a symmetrical rigid function. They work out the theory for the small bending of a thin incompressible plate initially stretched by uniform stresses in its plane. Finally, they generalize this theory to the case when small forces act in the plane of the plate.

C. Truesdell (Bloomington, Ind.).

Cicala, P. The effect of initial deformations on the behaviour of a cylindrical shell under axial compression. Quart. Appl. Math. 9, 273–293 (1951).

The general, non-linear equations for the radial displacement and stress function of a thin cylindrical shell with initial deformations, previously given by the author [Aerotecnica 24, 3–18 (1944)] are stated. It is the purpose of the investigation to determine whether initial deformations can be introduced to cause buckling to follow a favorable path (increasing load with increasing deformations). Approximate solutions are obtained by using the Galerkin procedure to determine a finite number of coefficients in the approximate forms of the stress function and the radial displacement. It is shown that for a cylindrical shell under axial compression buckling will in general follow an unfavorable mode regardless of the initial deformations. A cylindrical panel subject to (a) constant axial shortening and (b) constant axial stress is also studied. It is shown that in this case it is possible for the critical load of the linear theory to be exceeded before the beneficial effects of certain initial deformations are nullified. An approximate analysis shows that these favorable initial deformations may be obtained by bending the panel to a radius which is smaller than that at which it is tested. *G. H. Handelman.*

Reissner, Eric, and Stein, Manuel. Torsion and transverse bending of cantilever plates. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 2369, 39 pp. (1951).

The total energy for a thin plate of variable thickness and plan form including normal loading, loading in the plane, tip forces, and inertial loadings is set up. A deflection function of the form $w=W(x)+y\theta(x)$ is introduced and the corresponding Euler equations yield ordinary differential equations in $W(x)$ and $\theta(x)$. This may be regarded as taking the first two terms of a Taylor series development in y of $w(x, y)$. The following problems are considered: (A) rectangular plate of constant thickness under tip torque, uniform distribution of applied twisting moments, torsional vibrations, lateral buckling; (B) symmetrical plan form plate with chord and thickness variation with symmetric cross-section and spanwise variation of chord and thickness according to a power law, and a rectangular section with constant chord and exponential spanwise variation in thickness; (C) skewed plate of constant thickness and chord under tip loading and uniform lateral loading. Group (A) can be compared with other solutions and in particular with elementary beam theory. *G. H. Handelman.*

Conway, H. D. Axially symmetrical plates with linearly varying thickness. J. Appl. Mech. 18, 140–142 (1951).

Starting with L. Föppl's analogy between the equation for the symmetrical bending of circular plates and the radial deformation of a rotating disk of variable thickness, the author employs solutions obtained by R. Gran Olsson [Ing.-Arch. 8, 270–275 (1937)] for disks in which the thickness distributions are of the form $h=h_{\max}(1-\rho)^m$ where ρ is a dimensionless radial variable. The complementary functions are hypergeometric functions but are expressible in simple closed form for m equal to the reciprocal of Poisson's ratio. The author carries over these solutions to the plate problem. For Poisson's ratio equal to 1/3, the corresponding variable thickness becomes linear. Since the complementary solutions are known, particular solutions can be found by variation of parameters. Examples are given and comparisons drawn with plates of constant thickness.

G. H. Handelman (Pittsburgh, Pa.).

Ōhasi, Yosio. Bending of a thin elliptic plate of an orthotropic material under uniform lateral load. Z. Angew. Math. Physik 3, 212–224 (1952).

The author extends the circular plate solution of Okubo [J. Appl. Phys. 20, 1151–1154 (1949); these Rev. 11, 485] to include elliptic plates. Solutions are given for plates with clamped and simply supported edges, the former having the same form as the well known closed form solution for the isotropic case. The simply supported solution is obtained with the use of curvilinear coordinates. Numerical results are plotted for oak plates. *H. D. Conway.*

Kucharski, W. Beiträge zur Theorie der durch gleichförmigen Schub beanspruchten Platte. I. Ing.-Arch. 18, 385–393 (1950).

This paper is the first of a series of three [see the two following reviews] concerned with the transverse motion

of a thin plate subject to uniform shear forces in the plane of the plate. The shear force may vary with time but such variation is sufficiently slow that rotational inertia can be neglected. The equation for the deflection in dimensionless variables is $w_{tt}+\Delta\Delta w+\theta w_{xy}=0$ where θ is a measure of the shear force. Part I of the series considers the case $\theta=$ const.

A solution of the form $w=\sum_i T_i(t)U_i(x, y)$ is considered. The functions U_i satisfy equations of the form

$$(*) \qquad \Delta\Delta U+\theta U_{xy}-\lambda U=0.$$

A remark is made about the existence of solutions of (*) based on Green's functions to be found from the corresponding static problem. There is a solution of (*) in the form $\lambda=a+b\theta$ (a, b constant) but it is shown that this will not fulfill the usual boundary conditions. A perturbation scheme of the form $\lambda=\lambda_0+\lambda_1\theta+\lambda_2\theta^2+\cdots$ with a similar series for U is applied. Detailed calculations are given for the square plate simply supported on all four sides. For this case $\partial\lambda/\partial\theta=0$ for $\theta=0$, although this does not hold for other rectangular plates. A series is given for the gravest eigen value. The corresponding critical shear stress, obtained by setting $\lambda=0$, agrees closely with previously found values.

G. H. Handelman (Pittsburgh, Pa.).

Kucharski, W. Beiträge zur Theorie der durch gleichförmigen Schub beanspruchten Platte. II. Ing.-Arch. 18, 394–409 (1950).

Part II of this series is concerned with the vibrations of a plate under shear forces varying harmonically in time. The differential equation, $w_{tt}+\Delta\Delta w+(\Theta+\theta_0\sin\omega t)w_{xy}=0$, is first investigated by a perturbation expansion in θ_0. The zero-th order term w_0 is the case treated in part I. Stability conditions for each component in the series are developed. These depend on the frequency of the time component in w_0. Formulas are given for perturbations up to and including θ_0^4. Specific numerical calculations for a square plate in the lowest mode and θ_0^2 small are given. A stability map of the type appearing in the study of Mathieu's equation is developed. The author points out that the complete solution will require an atlas of infinitely many such maps, one for each frequency in w_0. A solution by Fourier series methods is indicated and applied to the problem of part I. These methods lead to infinitely many differential equations in infinitely many unknowns. *G. H. Handelman.*

Kucharski, W. Beiträge zur Theorie der durch gleichförmigen Schub beanspruchten Platte. III. Ing.-Arch. 19, 22–30 (1951).

The author continues his investigation [see the two preceding reviews] of the motion of a plate under uniform shear by treating the basic equation as one of wave motion. As in part I, the shear force does not depend on time. He finds for plane waves that the wave velocity depends on wave length, applied shear stress, and the angle of the wave front. Standing wave solutions for parallel strips with simply supported edges are found for suitable boundary forces. Similar solutions can be determined for certain types of parallelograms. A rhombus and certain rectangles appear as special cases. The difficulties of generalizing this approach are pointed out. *G. H. Handelman* (Pittsburgh, Pa.).

Bloh, V. I. Stresses in plane curved beams bounded by two intersecting arcs of circles for several cases of loading. Akad. Nauk SSSR. Inženernyĭ Sbornik 6, 47–54 (1950). (Russian)

The boundaries of the beam described as in the above title are circular arcs of different, completely arbitrary curvatures. Similar problems have been solved before by J. H. Michell [Proc. London Math. Soc. 32, 35–61 (1901)] for a beam bounded by arcs of the same curvature and one specific case of loading, by H. Neuber [Ing.-Arch. 6, 325–334 (1935)] and by P. Fillunger [Z. Angew. Math. Mech. 10, 218–227 (1930)] for a circular segment. This work is much more general and embraces the cases of the previous authors. Five cases of loading are considered. (a) Two opposite forces acting in the line of the common chord, applied at the ends. (The ends are the intersections of the bounding arcs.) (b) A constant distributed load normal to one of the circular boundaries, the ends of the beam supported. (c) A constant distributed load normal to one of the circular boundaries, one end of the beam fixed. (d) A concentrated force normal to the common chord applied at one end, the second end fixed. (e) Two couples applied at the ends. The author gives ready formulas for the stresses without showing the actual solution. He gives, however, a detailed description of how the solutions were obtained. He utilizes a modified Airy stress function which he calls the moment function. He uses also the method of conformal transformation by inversion. *T. Leser* (Lexington, Ky.).

✓ ***Clark, R. A., and Reissner, E. Bending of curved tubes.** Advances in Applied Mechanics, vol. 2, edited by Richard von Mises and Theodore von Kármán, pp. 93–122. Academic Press, Inc., New York, N. Y., 1951. $6.50.

The bending of a curved tube is considered as a problem in the theory of axisymmetrical stress distributions in a shell of revolution for which the circumferential component of displacement is not axisymmetric. Previous analyses of this problem have been based on variational principles.

The basic differential equations are established for the tube of circular cross-section and the solution expressed in terms of trigonometric series. An infinite set of linear equations in the coefficients results. Approximations are obtained by taking a finite number of terms. One such method reproduces results obtained by the principle of least work whereas the second includes the minimum potential energy results of von Kármán [Z. Verein. Deutsch. Ingenieure 55, 1889–1895 (1911)]. A second type of solution is found by determining the asymptotic form of a particular integral of a nonhomogeneous linear differential equation by an extension of the W. K. B. J. method. The parameter in question is $\mu=[12(1-\nu^2)]^{1/2}b^2/(ah)$ where ν is Poisson's ratio, b is the radius of the cross-section of the tube, a is the radius of the center line, and h is the thickness of the tube. The asymptotic solution is compared with the trigonometric solution and Beskin's [J. Appl. Mech. 12, A1–A7 (1945)] extensive computations. It is seen that the asymptotic solution is also valid for small values of μ. The asymptotic solution for a tube with uniform elliptical cross-section is also developed. *G. H. Handelman* (Pittsburgh, Pa.).

Malkina, R. L. On the theory of thin-walled curvilinear bars. Akad. Nauk SSSR. Inženernyĭ Sbornik 9, 201–208 (1951). (Russian)

The theory of thin-walled curvilinear bars given by V. Z. Vlasov [in a chapter added to the Russian edition of S. P. Timoshenko, The Stability of Elastic Systems, Gostehizdat, 1946] is an extension of Vlasov's theory of straight bars [Thin-walled Elastic Bars, Gosstrolizdat, 1940]. Vlasov assumed that the equilibrium equations (the six Kirchhoff equations) and the relations between the generalized forces and the displacements for straight bars are also valid for curvilinear bars with some small corrective terms. He did

not justify his assumption. G. Yu Džanelidze [Akad. Nauk SSSR. Prikl. Mat. Meh. 8, 25–32 (1944); these Rev. 6, 252] also extended the theory of straight bars to curvilinear bars. His method differs somewhat from that of Vlasov. The author of this paper in developing the theory of curvilinear bars follows Džanelidze rather than Vlasov. She finds also the conditions for which the Kirchhoff equilibrium equations are applicable to thin-walled bars. The author's results agree with those of Džanelidze, and differ a little from those of Vlasov. *T. Leser* (Lexington, Ky.).

Šerman, D. I., and Narodeckiĭ, M. Z. On the torsion of some prismatic hollow bodies. Akad. Nauk SSSR. Inženernyĭ Sbornik 6, 17–46 (1950). (Russian)

The authors consider two prismatic hollow bars with the following cross-sections: 1) an ellipse with a concentric circular hole, and 2) a square with rounded edges with a concentric circular hole. In both cases the bars are twisted by a moment M acting on the outside contour. The problems are solved by conformal mapping, which, due to the fact that the cross-section is a doubly connected region, presents considerable difficulty. The method employed was developed by Šerman [Doklady Akad. Nauk SSSR 63, 499–502 (1948); these Rev. 10, 651]. The obtained stress function, and hence the expressions for the stresses are in a form of complicated series. To make them useful for practical applications the authors tabulate for both cases up to twelve coefficients of the series for several dimension ratios. *T. Leser.*

Ling, Chih-Bing. Torsion of a circular cylinder having a spherical cavity. Quart. Appl. Math. 10, 149–156 (1952).

This paper provides a solution of the torsion problem for an infinitely long circular cylinder having a symmetrically located spherical cavity. The author constructs a solution $\Psi_0(r, z)=\Psi_0(r)+\Psi(r, z)$ of the Michell-Föppl differential equation such that Ψ_0 is the solution for a solid circular shaft and Ψ_1 is an auxiliary solution which vanishes on the surface of the cylinder and is an infinite sum of Legendre associated functions of the first kind of degree n and order 2. To require that $\Psi(r, z)$ vanish on the surface of the inner spherical cavity, the author is required to solve a linear infinite system for the proper coefficients in the infinite sums. There is considerable tedious calculation necessary to show the distribution of shear stress. It appears that the maximum shear stress always occurs on the external boundary across the minimum section. *D. L. Holl* (Ames, Iowa).

Šeremet'ev, M. P. The influence upon a homogeneously stressed plane field of an elastic ring welded into a curvilinear opening. Ukrain. Mat. Žurnal 1, no. 3, 68–80 (1949). (Russian)

Melan, E. Wärmespannungen in einer Scheibe infolge einer wandernden Wärmequelle. Ing.-Arch. 20, 46–48 (1952).

The process of welding a straight seam in a sheet of metal leads to the problem of the determination of the stresses in a thin disc of infinite radius caused by a point source of heat that moves along a radial line. It is known that the temperature distribution T and the stress potential θ satisfy the equations

$$\Delta T=-2\lambda T_x, \quad \Delta\theta=\frac{m+1}{m-1}\alpha T, \quad \text{where} \quad \Delta\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.$$

Using the known solution of the first equation, the author shows that it is possible to express the stresses σ_{xx}, σ_{yy}, σ_{xy} in terms of Hankel functions of the first kind without actually solving the second equation. A pictorial representation of the principal stress lines and a table of values of the principal stresses are given. *C. G. Maple* (Washington, D. C.).

Eschler, H. Zur Ermittlung der Eigenschwingungszahlen der in ihrer Mittelebene belasteten Rechteckplatte. Ing.-Arch. 18, 330–337 (1950).

This paper is concerned with determining the natural frequencies of transverse vibrations of a rectangular plate under constant normal stresses in the middle plane. Exact solutions are found for the cases where (1) two sides are simply supported and two sides clamped, and (2) two sides are simply supported and two sides free. One approximation for (1) is based on the work by Sezawa [Z. Angew. Math. Mech. 12, 275–279 (1932)] on vibrations of beams. A second type of approximate solution is found by a Rayleigh principle using product representations for the deflection function. These functions of one variable are the solutions of the corresponding beam problems. Problems (1) and (2) are studied by this method as well as the case of a plate clamped on all four sides. Close agreement is found between the exact solutions and the approximate solutions. The critical buckling loads can be found from these solutions by considering the solutions for vanishing natural frequencies.

G. H. Handelman (Pittsburgh, Pa.).

Caligo, Domenico. Complementi analitici e numerici allo studio delle aste vibranti. I, II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 12, 76–83, 277–284 (1952).

The author tabulates analytical and numerical data relevant to the use of variational methods in analyzing vibrations of beams with nonuniform section characteristics.

F. B. Hildebrand (Cambridge, Mass.).

Sato, Yasuo. Study on surface waves. III. Love-waves with double superficial layer. Bull. Earthquake Res. Inst. Tokyo 29, 435–444 (1951). (English. Japanese summary)

Pour les deux premières parties voir le même Bull. 29, 1–11, 223–261 (1951); ces Rev. 13, 705. L'auteur affirme, d'après des exemples numériques malheureusement assez particuliers, que la condition (nécessaire et suffisante?) d'existence des ondes de Love dans un milieu élastique indéfini surmonté de deux couches est que la vitesse des ondes transversales dans le milieu ne soit pas la plus petite des trois. Il montre en outre qu'il n'y a pas de discontinuité de la vitesse des ondes de Love lorsque leur vitesse passe par celle des ondes transversales dans la couche intermédiaire.

J. Coulomb (Paris).

Sato, Yasuo. Study on surface waves. IV. Equivalent single layer to double superficial layer. Bull. Earthquake Res. Inst. Tokyo 29, 519–528 (1951). (English. Japanese summary)

Dans un milieu élastique indéfini surmonté d'une couche unique et dans un milieu analogue surmonté de deux couches les courbes de dispersion des ondes de Love peuvent être très approximativement les mêmes si les constantes numériques des deux cas sont convenablement choisies.

J. Coulomb (Paris).

Ziegler, H. Die Stabilitätskriterien der Elastomechanik. Ing.-Arch. 20, 49–56 (1952).

The author shows that in the case of systems that are not conservative certain of the commonly used criteria for stability are not applicable. *H. W. March* (Madison, Wis.).

Hoff, N. J. The dynamics of the buckling of elastic columns. J. Appl. Mech. **18**, 68–74 (1951).

The author points out that the classical column buckling load is derived from consideration of equilibrium under constant loads, whereas the actual test is a dynamic process in which the column fails while the load is being increased. He considers the motion of a column of uniform cross section having a slight initial deflection in a testing machine with head moving at uniform velocity. The resulting non-linear partial differential equation can be reduced to the study of a non-linear ordinary differential equation with a non-dimensional time appearing as independent variable. If the initial deflection and total deflection are small, the equation may be linearized and leads to two forms of Bessel function solutions depending on whether the displacement of the head is less than or greater than the Euler displacement (static buckling strain multiplied by the column length). A solution in power series is given when the non-linear terms cannot be neglected. For large values of the non-dimensional time, it can be shown that the dynamic deflection curve can be represented as small oscillations superimposed on the static deflections. Numerical examples are given.

It is also shown that two columns are dynamically similar in such a process for small deflections if the buckling index $\Omega=\pi^3(a/c)^2(\rho/L)^4$ [a/c = velocity of sound/velocity of head, ρ/L slenderness ratio] is the same. Dynamic similarity for large deflections requires the initial deviation from straightness to be the same. *G. H. Handelman.*

Nowacki, W. Sur les problèmes de la stabilité d'une plaque orthotrope. Arch. Méc. Appl., Gdańsk **2**, 169–182 (1950). (French. Polish summary)

The buckling of an infinitely long orthotropic plate under pairs of concentrated compressive forces perpendicular to the long edges is studied by means of an approximation due to Sommerfeld [Z. Math. Physik **54**, 113–153 (1907)]. The method consists in dropping the initial stress terms from the differential equations and adding a boundary condition which is equivalent to the equilibrium of the transverse forces across a section of the plate at the point where the concentrated forces are applied. The equation is solved by separation of variables and the critical loads computed from the boundary conditions. The following cases are considered: a single pair of concentrated forces, an infinite set of equal concentrated forces equally spaced, a pair of forces at the edge midpoints of a finite rectangular simply supported plate, concentrated forces with sinusoidal distributed forces acting between them, a clamped rectangular plate, two pair of concentrated forces, and a semi-infinite plate under one pair of forces. *G. H. Handelman* (Pittsburgh, Pa.).

***Plantema, Frederik Johan. Theory and experiments on the elastic overall instability of flat sandwich plates.** Thesis, Technische Hogeschool te Delft. Eduard Ijdo, Leiden, 1952. viii+67 pp.

In a previous paper [Proc. Seventh Internat. Congress Appl. Mech. v. 1, 200–213 (1948); these Rev. **11**, 558] the author, independently of L. H. Donnell [J. Appl. Mech. **13**, A-249–A-252 (1946); these Rev. **8**, 241], used a method of partial deflections to evaluate the effect of shear deformation in the core of flat sandwich plates. In this method the deflection of the plate is considered to be made up of two parts, one associated with the bending moments, the other with the shear deformation in the core. In the present paper a critical study is made of the general applicability of the method. The basic derivations follow those of the paper by Libove and Batdorf [Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1526** (1948); Tech. Rep. ibid., no. **899** (1948); these Rev. **10**, 86; **11**, 486]. For sandwich plates composed of isotropic materials the method is found to be rigorously applicable if the edges are simply supported. It is applied to the problems of the compressive buckling of rectangular plates and the shear buckling of infinitely long plates, the edges being simply supported in both cases. A method is presented for taking into account the flexural rigidities of the facings. For the case of an infinitely long plate with clamped edges under longitudinal compressive loading it is found in using the method of partial deflections that one of the three boundary conditions cannot be satisfied. However, the effect of this condition appears to be small. In general for orthotropic sandwich plates the resolution of the deflection into two components is no longer possible. The resolution must be into two components in each of the directions of the orthotropic axes. Complications are thus introduced. The only case for which a solution is presented is that of the compressive buckling of an infinitely long orthotropic plate with simply supported edges. Comparison is made throughout the paper of the results of the method of partial deflections with the method of split rigidities of P. P. Bijlaard [Nederl. Akad. Wetensch., Proc. **50**, 79–87, 186–193 (1947); these Rev. **9**, 164]. The results are given of a series of compressive buckling tests of sandwich plates with simply supported edges. The experimental buckling loads were much less than those predicted by theory. This fact is attributed to initial lack of flatness of the plates tested and to other possibilities. *H. W. March* (Madison, Wis.).

Eringen, A. Cemal. Buckling of a sandwich cylinder under uniform axial compressive load. J. Appl. Mech. **18**, 195–202 (1951).

To study the instability of a sandwich-type airplane fuselage, the author studies the buckling of a sandwich cylinder under uniform axial compressive load in the following manner. Since the core material is much softer than the faces, the compressive load is assumed to be carried by the faces. The core is treated as an elastic tube with zero longitudinal stress and radial displacements at its ends but arbitrary deflections on the lateral surface. The equilibrium problem for the core is then solved by separation of variables in terms of infinite series. The deflections of the faces are the surface displacements of the core. Hence the buckling equation for the faces can be set up in terms of the surface displacements and stresses found from the core equilibrium problem. The resulting critical values are found and simplifying formulas for large radii are given. Numerical and graphical results for several types of materials are given. *G. H. Handelman* (Pittsburgh, Pa.).

Lebedev, N. F. On the propagation of unloading waves in the case of linear hardening. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 625–628 (1951). (Russian)

In a previous paper Rakhmatulin [same journal **9**, 91–100 (1945); these Rev. **7**, 144] discussed the propagation of an impact elastic-plastic wave in a homogeneous semi-infinite rod for the case in which the end loading can be described by an analytic function. The present paper gives an approximate method for determining stress in the rod produced by an arbitrary end loading by use of a net of characteristics of the propagation equations. The solution is given in the form of a converging sum. This is applied to the case of linear hardening. *H. I. Ansoff* (Santa Monica, Calif.).

Mežlumyan, R. A. Calculation of the load bearing capacity of thin-walled construction taking account of hardening of the material. Akad. Nauk SSSR. Prikl. Mat. Meh. 15, 175–182 (1951). (Russian)

In two previous papers [same journal 14, 253–264, 537–542 (1950); these Rev. 12, 373; 13, 406] the author developed equations for determination of the state of stress in a cylindrical shell produced by prescribed loadings. The present paper applies these results to the problem of determining the load-bearing capacity of thin rods and cylindrical shells which exhibit hardening. *H. I. Ansoff.*

Birger, I. A. Some general methods of solution of problems of the theory of plasticity. Akad. Nauk SSSR. Prikl. Mat. Meh. 15, 765–770 (1951). (Russian)

It is shown that solution of an elastic-plastic problem is equivalent to solution of one of three elastic problems: 1) elastic problem with the same strain distribution as the plastic problem and additional loading; 2) elastic problem with the same stress distribution and an additional strain distribution; 3) elastic problem with same stress and strain distributions and variable elastic constants. Iterative techniques are described for solution of each of these equivalent problems. *H. I. Ansoff* (Santa Monica, Calif.).

Šapiro, G. S. Elastic-plastic equilibrium of a wedge and discontinuous solutions in the theory of plasticity. Akad. Nauk SSSR. Prikl. Mat. Meh. 16, 101–106 (1952). (Russian)

The statically determinate problem of a wedge is solved for regimes of elastic-plastic stress distributions. In limiting cases the solutions are shown to reduce to the wholly plastic regimes obtained by Prager. It is pointed out that for a wholly plastic sharp wedge it is necessary to select from the family of possible statically determinate solutions that which also satisfies the velocity boundary conditions and the appropriate stress-strain relations. *H. I. Ansoff.*

Panferov, V. M. On the convergence of a method of elastic solutions for the problem of elastic-plastic bending of a plate. Akad. Nauk SSSR. Prikl. Mat. Meh. 16, 195–212 (1952). (Russian)

A method of solution of the problem of elastic-plastic bending of thin plates was proposed by Ilyushin which employs iteration of equivalent elastic solutions. The present paper transforms the integro-differential equations of the problem into a system of nonlinear integral equations and uses certain topological theorems to prove existence of a solution and hence convergence of the iterative procedure. *H. I. Ansoff* (Santa Monica, Calif.).

Grigor'ev, A. S. On the bending of a round elastic plate beyond the elastic limit. Akad. Nauk SSSR. Prikl. Mat. Meh. 16, 111–115 (1952). (Russian)

Consider a simply supported circular plate exhibiting linear hardening and loaded uniformly over a circle concentric with the plate. At a critical value of the load plastic deformations will first appear at the center on the underside of the plate. The paper presents a closed form solution for the elastic part and a numerical procedure for determination of the plastic region. Results of a special experiment are compared with a numerical solution and good agreement is found. *H. I. Ansoff* (Santa Monica, Calif.).

Ševčenko, K. N. An axisymmetric elastic-plastic problem for a plate weakened by a circular cut-out. Akad. Nauk SSSR. Prikl. Mat. Meh. 15, 519–520 (1951). (Russian)

The surface of a circular hole in an infinite thin plate is deformed beyond the elastic limit by a hydrostatic pressure. The material is assumed to be incompressible and to exhibit linear hardening. Under these conditions a solution is given for the stress and strain distributions in the plastic region. *H. I. Ansoff* (Santa Monica, Calif.).

Ševčenko, K. N. Plane elastic-plastic deformation of a cylinder loaded by a balanced system of two concentrated forces. Akad. Nauk SSSR. Prikl. Mat. Meh. 16, 35–44 (1952). (Russian)

Author's summary: "It is known from experiments that compression of a sphere between two plane-surfaced plungers produces cracks at the center, while rolling produces cavities. The present paper presents an effort to explain the quite complicated mechanism of formation of cavities by means of an approximate model. A more detailed study of this phenomenon will require a great deal of work. Here we consider the plane problem of distribution of elastic and plastic deformations in a section of the cylinder subjected to the action of two concentrated forces. In the plastic region an approximate solution is obtained under the assumption of incompressibility of the material." *H. I. Ansoff* (Santa Monica, Calif.).

Rabotnov, Yu. N. An approximate engineering theory of elastic-plastic shells. Akad. Nauk SSSR. Prikl. Mat. Meh. 15, 167–174 (1951). (Russian)

An approximate theory is proposed which applies under the conditions of zero curvature in one direction along the shell and no applied force in the other direction. In the plastic region the shell is represented by two parallel layers and the spacing between the layers is determined so as to satisfy plastic-elastic boundary conditions. A number of examples is given in which the corresponding shell equations are reduced to somewhat simpler non-linear forms. Numerical integration procedures are briefly discussed. *H. I. Ansoff* (Santa Monica, Calif.).

Koval'skiĭ, B. S. The elastic-plastic bending of a beam on an elastic foundation. Doklady Akad. Nauk SSSR (N.S.) 77, 209–211 (1951). (Russian)

A method is proposed for determination of the moment distribution in a beam exhibiting linear hardening which is stressed beyond the elastic limit. The problem is reduced to solution of a fourth order nonlinear equation. *H. I. Ansoff* (Santa Monica, Calif.).

Bulygin, V. Ya. On elastic-plastic torsion of prismatic rods. Akad. Nauk SSSR. Prikl. Mat. Meh. 16, 107–110 (1952). (Russian)

Author's summary: "This note considers the problem of elastic-plastic torsion of prismatic rods using a semi-inverse method, namely, a contour is determined for the cross-section of the rod which corresponds to a prescribed form of the elastic kernel." *H. I. Ansoff* (Santa Monica, Calif.).

Lazutkin, D. F. Propagation of elastic-plastic waves along a cylindrical rod. Akad. Nauk SSSR. Prikl. Mat. Meh. 16, 94–100 (1952). (Russian)

Author's summary: "The paper presents the case of propagation of elastic-plastic waves in a cylindrical homogeneous rod. The pressure on the left end rises according

to an arbitrary law and then suddenly drops to zero. The hardening is nonlinear and can be represented by a convex curve. It is assumed that the rod is long enough so that the reflections from the right end of the rod need not be considered." *H. I. Ansoff* (Santa Monica, Calif.).

Lepik, Yu. R. Supplementary remarks on the cylindrical form of buckling of plates beyond the elastic limit. Akad. Nauk SSSR. Prikl. Mat. Meh. 15, 107–110 (1951). (Russian)

The problem of cylindrical loss of stability of a thin rectangular plate sufficiently long in one direction and uniformly compressed in the other was solved by Ilyushin [Plasticity . . . , Moscow-Leningrad, 1948; these Rev. 12, 373] and Popov [same journal 14, 543–552 (1950); these Rev. 13, 94]. However, both of these authors assumed that the material is incompressible and that the plastic deformations just before loss of stability are negligibly small. The present paper generalizes the earlier results by removing these two restrictions. It is shown that the incompressibility assumption introduces no serious error into the results. On the other hand, the assumption of negligible plastic deformations may result in an appreciable error in the solution.

H. I. Ansoff (Santa Monica, Calif.).

Lepik, Yu. R. Buckling of plates of a compressible material in the plastic range. Akad. Nauk SSSR. Prikl. Mat. Meh. 15, 629–634 (1951). (Russian)

The loss of stability of a plate in the plastic state behaving according to the von Mises condition is investigated under the following conditions: 1) The body forces are zero, and 2) variations in the normal and shear forces are zero. It is shown that during loss of stability the plate can remain wholly plastic only if the material is incompressible. The effect of compressibility on the critical stiffness is next estimated. It is shown for six types of plate heading that, for $\nu=0.3$, assumption of incompressibility leads to an error not exceeding 15.5%. *H. I. Ansoff.*

Kačanov, L. M. The stability of the plane form of bending beyond the elastic limit. Doklady Akad. Nauk SSSR (N.S.) 77, 989–992 (1951). (Russian)

The stability of a plane strip subjected to bending is discussed using the theory of small elastic-plastic deformations. The lateral force, the bending moments, and distortion of the plane sections of the strip are all assumed to be small. Under these conditions it is shown that the resulting stability problem is represented by linear homogeneous equations with variable coefficients and homogeneous boundary conditions. It pointed out that, consequently, plane bending beyond the elastic limit does not necessarily entail loss of stability. *H. I. Ansoff.*

Kačanov, L. M. Stability of the plane form of bending beyond the elastic limit. Akad. Nauk SSSR. Prikl. Mat. Meh. 15, 195–206 (1951). (Russian)

This is an expanded version of the paper reviewed above. The theory of small elastic-plastic deformations is applied to the problem of stability of a thin rectangular rod in simple bending. The expressions for strains are assumed to have the same form as in the elastic problem and it is shown that a cross-section of the rod will, in general, have a rectangular elastic region, a region of plastic loading, and a region of unloading, the latter two being separated by a straight line through the centroid of the section. Equations are derived relating the three regions. Next, equations are derived which in effect introduce variable stiffness terms into Kirchhoff's equation for elastic bending. The stability problem is thus reduced to a system of homogeneous linear equations with variable coefficients. An energy method is discussed and the results applied to several examples to obtain curves of the critical bending moment. *H. I. Ansoff.*

Kačanov, L. M. Stability of the plane form of bending beyond the elastic limit (influence of hardening). Akad. Nauk SSSR. Prikl. Mat. Meh. 15, 637–641 (1951). (Russian)

In the paper reviewed above the author formulated the problem of stability of a rectangular rod in flexure beyond the elastic limit and applied it to ideally plastic materials. The present paper applies this formulation to materials with linear hardening. It is shown that under the limiting condition the resulting equations reduce to the ideally plastic and to the elastic cases respectively.

H. I. Ansoff (Santa Monica, Calif.).

Kačanov, L. M. Stability of the plane form of bending beyond the elastic limit. III. Influence of complex loading. Akad. Nauk SSSR. Prikl. Mat. Meh. 15, 762–764 (1951). (Russian)

It is pointed out that the theory of small elastic-plastic deformations is not strictly applicable to this problem, since the loading is "complex" because the stress components do not increase proportionately to a given parameter. The theory of plastic flow is applied and the solution is shown to be essentially in agreement with previous solutions by the author obtained by employing the former theory [cf. the three preceding reviews]. *H. I. Ansoff.*

Udeschini, Paolo. La plasticità nella meccanica aleatoria. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 13(82), 255–266 (1949).

The author considers the basic plasticity equations consisting of: (1) the equations of motion; (2) the continuity relation; (3) the yield condition of von Mises (expressed in terms of the rate of strain tensor by use of a linear relation between the over-stress tensor and the rate-of-strain tensor). Secondly, the equations of turbulence due to Dedebant and Wehrlé [Portugaliae Phys. 1, 95–149 (1944), 179–296 (1945); these Rev. 7, 129, 315] are introduced. These consist of the equations of motion and the continuity relation in terms of probability density functions for the velocity, density, and acceleration. In particular, the transfer or inertia term is treated as a stress and as in viscous fluid theory is divided into a mean stress (the pressure), a term due to the viscous stresses, and a compressible term. By assuming the existence of a relation corresponding to the von Mises yield condition, the author obtains an equation involving the dissipation function, a constant (corresponding to the plastic yield constant), and a variable corresponding to the constant in Hooke's law. From dimensional analysis, it follows that a function of dimension time exists which is inversely proportional to the absolute temperature and the dissipation function. By comparing this formula with a result of Dedebant and Wehrlé, the author concludes that the accumulative velocity probability function of statistical mechanics is inversely proportional to the square of the temperature. *N. Coburn* (Ann Arbor, Mich.).

Jaswon, M. A., and Foreman, A. J. E. The non-Hookean interaction of a dislocation with a lattice inhomogeneity. Philos. Mag. (7) **43**, 201–220 (1952).

The authors elaborate a procedure for calculating the external shear stress τ required to hold edge dislocations of Peierls-Nabarro type [Peierls, Proc. Phys. Soc. **52**, 34–37 (1940); Nabarro, ibid. **59**, 256–272 (1947)] in equilibrium at prescribed distances from certain types of lattice inhomogeneities. Their formulation of the problem of the interaction of dislocations with lattice inhomogeneities reduces the above calculation to the following question: to compute τ for a Peierls-Nabarro dislocation "when the atomic structure along the slip plane is no longer constant but varies in a specified manner". Mathematically speaking, this is equivalent to solving a certain non-linear integral equation for the shear stress along the slip plane of interest.

They consider the following physically interesting problems involving a single dislocation of this type: (i) its interaction with a Zener micro-crack [Zener, Physical Rev. (2) **69**, 128–129 (1946)]; (ii) with a grain boundary; (iii) with an inhomogeneity consisting of a layer of crystalline material with different lattice parameters. They also formulate and solve approximately the problem of an array of collinear edge dislocations held immobile before a barrier by an appropriate external τ. *A. W. Sáenz.*

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Grosjean, C. C. Note on the diffraction of light by a finite number of centres distributed at random. Nuovo Cimento (9) **9**, 220–234 (1952).

Exact formulae are derived for the probability distribution of the amplitude and phase of the Fraunhofer diffraction field due to an arbitrary number of randomly distributed apertures in an opaque screen. Generalization of results due to Toraldo di Francia [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 186–189; **3**, 568–573 (1947); **4**, 730–735 (1948); these Rev. **10**, 274].

C. J. Bouwkamp (New York, N. Y.).

Groschwitz, E., und Hönl, H. Die Beugung elektromagnetischer Wellen am Spalt. I. Z. Physik **131**, 305–319 (1952).

The more general case of the theory of the diffraction of plane electromagnetic waves on a diaphragm [Z. Physik **131**, 290–304 (1952); these Rev. **13**, 801] is applied to the special case of the diffraction on a slit. The slit is located in the xy-plane and is bounded by the lines $x=\pm b$. The incident electromagnetic wave, with the electric vector parallel to the y-axis is represented by (a) $u_0=e^{ikz}$, $k=2\pi/\lambda$. The electromagnetic field is given by

$$u(x,z)=\int_{-\infty}^{\infty} g(\alpha)e^{ik(\alpha x+\gamma z)}d\alpha \tag{b}$$

in the upper half-plane $z\geqq 0$, and by

$$u(x,z)=e^{ikz}-e^{-ikz}+\int_{-\infty}^{\infty} g(\alpha)e^{ik(\alpha x-\gamma z)}d\alpha \tag{c}$$

in the lower half-plane $z\leqq 0$. The weight function $g(\alpha)$ satisfies the dual integral equation

$$\int_0^{\infty} g(\alpha)\cos(kx\alpha)d\alpha=0 \quad \text{for} \quad |x|\geqq b, \tag{d}$$

$$\int_0^1 [1-\alpha^2]^{\frac{1}{2}}g(\alpha)\cos(kx\alpha)d\alpha + i\int_1^{\infty}[\alpha^2-1]^{\frac{1}{2}}g(\alpha)\cos(kx\alpha)d\alpha=\tfrac{1}{2} \quad \text{for} \quad |x|\leqq b. \tag{e}$$

For the solution (d), (e) the try

$$g(\alpha)=\sum_{m=0}^{\infty} C_m\alpha^{-1}J_{2m+1}(kb\alpha) \tag{f}$$

is made. By virtue of the Weber-Sonine integral formula it is obvious, that (f) satisfies (d) term by term, and the C_m have to be determined such that (f) satisfies (e). For the case of a narrow slit ($kb\ll 1$) the first two coefficients of (f), C_0 and C_1, are evaluated. *F. Oberhettinger.*

Buchholz, Herbert. Der schleifenerregte Hohlraumresonator aus zwei konfokalen drehparabolischen Kappen. Arch. Elektr. Übertragung **6**, 6–16 (1952).

An electromagnetic field is excited in the interior of a cavity resonator which is bounded by two confocal (different) paraboloids of revolution with a common axis of revolution (z-axis of a Cartesian system of coordinates). The exciting source is a circularly shaped loop with a radius small enough to assure a uniform distribution of a current J along the loop. The axis of the loop coincides with the axes of revolution of the paraboloids. Introducing parabolic coordinates (ξ, η, φ), it is shown that the only nonvanishing electric component E_φ satisfies the differential equation

$$\frac{\partial}{\partial\xi}\left(\xi\frac{\partial E_\varphi}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\eta\frac{\partial E_\varphi}{\partial\eta}\right)-\frac{\xi+\eta}{4\xi\eta}E_\varphi+(\xi+\eta)k^2E_\varphi=0.$$

A solution of this differential equation which satisfies the boundary conditions (vanishing of the tangential electric components on the boundary of the cavity resonator) is given in the form of a contour integral involving Whittaker's function of the first and second kind. This integral is expanded into eigenfunctions of the paraboloid of revolution. The given results can be simplified if the cavity resonator is symmetrical. For this case a formula for the resonance frequencies is given and the first 6 are computed and compared with those of a spherical cavity resonator of a) equal volume and b) equal surface. *F. Oberhettinger.*

Goldoni, Gino. Sulla propagazione delle onde elettromagnetiche in un cavo riempito da un dielettrico eterogeneo. Atti Sem. Mat. Fis. Univ. Modena **4**, 105–121 (1950).

The author considers electromagnetic oscillations in the space between two coaxial circular cylinders on the assumption that the dielectric constant of the medium in the cavity is a variable function of the distance from the axis. His work is thus an extension of earlier work by D. Graffi on rectangular guides filled with nonhomogeneous dielectric [Ann. Mat. Pura Appl. (4) **30**, 233–239 (1949); these Rev. **12**, 65], and the results obtained are of similar form; namely, upper and lower bounds for the critical frequencies and for the phase and group velocities. The discussion is limited to circularly symmetric waves of magnetic and electric type.

M. C. Gray (Murray Hill, N. J.).

Goldoni, Gino. Una notevole proprietà degli integrali dell'equazione differenziale della propagazione delle onde elettromagnetiche in tubi circolari riempiti di dielettrico eterogeneo. Atti Sem. Mat. Fis. Univ. Modena **4**, 122–125 (1950).

In the paper reviewed above the author obtained a differential equation for the lowest order waves of electric type,

$$\frac{d}{d\rho}\left[\frac{1}{\epsilon\rho}\frac{d(\rho y)}{d\rho}\right]+\frac{1}{\rho}\left(\mu\omega^2-\frac{\beta^2}{\epsilon}\right)\rho y=0,$$

which reduces to Bessel's equation when ϵ is constant. The present note discusses the solution of the general equation in series of powers of ρ and shows that only one of the two fundamental solutions, corresponding to $J_1(k\rho)$, is regular at the origin. *M. C. Gray* (Murray Hill, N. J.).

Goldoni, Gino. Nuove limitazioni per le frequenze delle onde elettromagnetiche in un cavo riempito da dielettrico eterogeno. Atti Sem. Mat. Fis. Univ. Modena **4**, 126–132 (1950).

A Prüfer transformation of the differential equation in the review above, and of the corresponding equation for waves of magnetic type, leads to alternative sets of upper and lower bounds for the critical frequencies in the coaxial cavity. *M. C. Gray* (Murray Hill, N. J.).

Goldoni, Gino. Sulla teoria dell'oscillatore di Hertz. Atti Sem. Mat. Fis. Univ. Modena **4**, 97–104 (1950).

The author discusses the energy equation for an oscillator formed by two equal and oppositely charged spheres joined by a perfectly conducting cylindrical wire. He assumes that the charge $Q(t)$ on one sphere is assigned in a finite interval $(-\tau, 0)$, and uses a Laplace transform method to obtain an explicit expression for the transform of Q in terms of known integrals. The remainder of the paper discusses the convergence of these integrals and the existence and uniqueness of the inverse transform. *M. C. Gray.*

Pratelli, Aldo M. I tensori coniugati del campo elettromagnetico. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **14(83)**, 191–207 (1950).

If F_{ij} is the electromagnetic tensor (six-vector) in a galilean space-time, then the author calls its conjugate (or dual) ${}^*F^{ij}$ the magnetoelectric tensor. He expresses the usual equations of the electromagnetic field in terms of the conjugates of the vectors and tensors involved, and discusses at some length various integral electromagnetic or magnetoelectric theorems. *H. S. Ruse* (Princeton, N. J.).

Gross, Wolf. Sul calcolo dell'induttanza nei sistemi elettrici composti da conduttori paralleli. Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. 320 = L'Energia Elettrica **28**, no. 9, 5 pp. (1951).

The computation of the induction parameters for a multilinear transmission line becomes laborious when the number of conductors is large. The author considers the case of two rows of parallel equidistant conductors (n and n, or n and $n-1$, respectively) and computes the mutual inductance of the conductors of one row with respect to those of the other, and the self inductance of the conductors of each row separately. Apparently he assumes that all current densities in the conductors are uniform and all current phases for the conductors of each row are equal.

The purpose of the paper is to derive asymptotic formulas for a simplified computation of such terms as $\sum_{s=1}^{n-1}(n-s)\ln s$, $\sum_{s=1}^{n-1}(n-s)\ln(s-x)$ and similar ones, which enter in the expressions of the inductances. Two asymptotic expressions are derived, one based on a formula of Euler-MacLaurin and the other on a Taylor expansion. The first is in terms of Bernoulli numbers, the second in terms of certain polynomials which are tabulated. *L. Vallese.*

Zinn, M. K. Network representation of transcendental impedance functions. Bell System Tech. J. **31**, 378–404 (1952).

A transcendental, meromorphic function, $F(p)$ which satisfies the following conditions: (1) $F(p)$ has no poles in $\mathrm{Re}[p]\geqq 0$, (2) $F(p)=\bar{F}(p)$, (3) $\mathrm{Re}[F(p)]\geqq 0$ on $\mathrm{Re}[p]=0$, (4) all poles of $F(p)$ are simple, (5) $F(p)=O(1)$, exactly, as $|p|\to\infty$ everywhere except at the poles, is shown to represent the driving point impedance or admittance of an electrical network constructed with lumped, linear, positive elements R, L, G, and C. Generally, the network will contain an infinite number of branches. It is stated that conditions (1)–(3) are necessary and it is shown that conditions (1)–(5) constitute a sufficient set. The elimination of functions having poles on the imaginary p axis as required by (1) is debatable and seems unnecessarily restrictive. To synthesize the network, one obtains the Mittag-Leffler expansion of $F(p)$ and then by virtue of condition (4) proceeds to realize individual terms in this expansion by a method practically analogous to that of Brune and Foster. Several illustrative examples, exhibiting the equivalent network representations of transmission lines and cavities are given. *R. Kahal.*

Dahr, Konstantin. On the mathematical analysis of an idealized multiplex electromagnetic machine. An introduction to the theory of electromachinery. Trans. Chalmers Univ. Tech. Gothenburg [Chalmers Tekniska Högskolas Handlingar] no. **112**, 116 pp. (1951).

A generalized rotating electric machine is considered with any number of windings placed at arbitrary angles on several concentric cylinders, each rotating with a different speed. Transformations of reference frames from the rotating coils to stationary axes are included and both transient and stationary states are analyzed. The study is restricted to constant-speed operation and to machines with smooth air-gap. *G. Kron* (Schenectady, N. Y.).

Quantum Mechanics

✓ ***Feynman, Richard P. The concept of probability in quantum mechanics.** Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 533–541. University of California Press, Berkeley and Los Angeles, 1951. $11.00.

This paper has two objects: First, to give an elementary account of the conceptualism of quantum mechanics as applied to elementary particles, showing the effects of the disturbance produced by observation, indeterminancy, interference, etc.; second, to show that the classical laws of probability (the Laplace laws, as the author calls them) are incorrect in quantum mechanics. In his first task the author contributes nothing new; in his second, he falls into a fundamental error (common, unfortunately, to some others who have treated the subject). His reasoning is as follows:

Electrons fron a point source S impinge on a screen having two symmetric slits (1) and (2) and, if they get through, register their presence on a further screen, parallel to the

first (as in Young's interference experiment). Assuming that a particular electron gets through, let P be its probability of being recorded at the point X of the second screen. Let P_1 be the probability of the similar event when slit (2) has been closed, P_2 that with (1) closed. Then it is known that $P \neq P_1+P_2$, since in quantum mechanics there is electron interference. The author regards this as proving that the law of total probability (supposed to give $P=P_1+P_2$) is wrong.

The fallacy is that total probability applies only when the events are mutually exclusive. In the present case this is only true if the classical picture is used, according to which the electron occupies a definite position at each time and moves through such positions; then it can be said that it gets from S to X either through (1) or (2): two mutually exclusive events. But in quantum mechanics it is in principle meaningless to say that an electron really has some position (stated or unstated) when no position-observation is made. Much less is it meaningless to say that it really moves, in the sense of occupying a succession of positions in the course of time, when no appropriate observations are made. All that can be said regarding the event of probability P_1 is that it consists of the electrons being recorded at X under the conditions of (1) open and (2) closed ("through the agency of having only (1) open"). And the only meaningful description of event P is that it is the recording of the electron at X under the conditions of both (1) and (2) being open ("through the agency of (1) and (2) open", not "either passing through (1) or passing through (2)"). There is no mutual exclusiveness and hence no violation of total probability.

At the end of the paper, the author gives the treatment of the Schrödinger theory he published recently [Rev. Modern Physics 20, 367–387 (1948); these Rev. 10, 224], in which probabilities are regarded as complex quantities which experience interference. What this amounts to is simply an attempt at picturing in terms of a probability quite different from the usual one the following situation which has been known since 1930: If to space-time one adjoins action s, the Schrödinger equation can be written, with increase in simplicity and without imaginaries, in five independent variables (x, y, z, t, s). Then the classical Hamilton-Jacoby equation can be regarded as obtained as a limiting case, using the physical language of interference. To the mathematician, these physical and quasi-physical pictures seem unnecessary: the classical equation is simply the equation of the characteristics of the 5-dimensional Schrödinger equation. *B. O. Koopman.*

+Lenzen, Victor F. Philosophical problems of the statistical interpretation of quantum mechanics. Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 567–579. University of California Press, Berkeley and Los Angeles, 1951. $11.00.

An expository account of the well known general conceptual principles of quantum mechanics, viewed with the object of studying their philosophical implications. The statement of these latter appears to be scarcely more than a paraphrase in philosophic terms of the physical account. *B. O. Koopman* (New York, N. Y.).

Dugas, René. Sur l'interprétation de la mécanique quantique à l'aide de variables cachées au sens de M. David Bohm. C. R. Acad. Sci. Paris 234, 1599–1601 (1952).

The author makes several remarks on Bohm's new interpretation of quantum theory [Physical Rev. (2) 85, 166–179, 180–193 (1952); these Rev. 13, 709, 710], and compares it to that of Hertz concerning classical mechanics. *A. W. Sáenz* (Bloomington, Ind.).

Costa de Beauregard, O. Quelques aspects de l'irréversibilité du temps dans la physique classique et quantique. Rev. Questions Sci. (5) 13, 171–199 (1952).

Pluvinage, P. Nouvelle famille de solutions approchées pour certaines équations de Schrödinger non séparables. Application à l'état fondamental de l'hélium. J. Phys. Radium (8) 12, 789–792 (1951).

To get an approximate solution of the Schroedinger equation is the case where the potential energy consists of two parts, the author considers two Schroedinger equations each involving one of these parts and takes as an approximate solution of the original equation the product of the solutions of the two latter equations. By using such an approximate solution as the trial function in a variational principle, the author calculates the energy of the ground state of helium to an accuracy which is quite good considering the amount of calculation involved. *N. Rosen* (Chapel Hill, N. C.).

Löwdin, Per-Olov. A note on the quantum-mechanical perturbation theory. J. Chem. Phys. 19, 1396–1401 (1951).

A simple stationary treatment of conventional quantum-mechanical perturbation theory is made possible by using a variational principle to derive a system of linear algebraic equations corresponding to the secular equation. No restrictions are imposed upon the Hamilton operator H; only a system of approximate eigenfunctions of H is assumed to be known. The unperturbed states are divided into two classes, and the representatives of one class are eliminated by an iteration process. A perturbation formula is thus obtained which gives the influence of one class of states upon the other in the final solution. This formula represents a generalization of the Schroedinger-Brillouin formula for the eigenvalue of a non-degenerate state; it can be applied to degenerate states, and is particularly useful in cases where the system consists of two independent parts in mutual interaction. The mathematical formalism developed in this paper is closely connected with the theory of bordered determinants. *E. Gora* (Providence, R. I.).

Park, David. The scattering theory of the Schrödinger equation. Amer. J. Phys. 20, 293–300 (1952).

Murai, Yasuhisa. On the path integral and its application. Progress Theoret. Physics 6, 762–771 (1951).

This paper is concerned with formal applications of the method introduced by Feynman [Physical Rev. (2) 84, 108–128 (1951); these Rev. 13, 410] for calculating with ordered operators and functionals. The author derives the general formal solution of the Dirac and the Klein-Gordon equations with an external electromagnetic field. He uses the "proper time" parametrization introduced by Fock [Phys. Z. Sowjetunion 12, 404–425 (1937)] and studied by Nambu [Progress Theoret. Physics 5, 82–94 (1950); these Rev. 11, 766]. The results are the same as those found in an intuitive way by Feynman, but with a greater precision of mathematical statement. For illustration he calculates in detail the case of a free electron satisfying the Dirac equation without external field, and also calculates to second order the effect of the electron interacting with its own self-field. *F. J. Dyson* (Ithaca, N. Y.).

Sakuma, Kiyoshi, Shôno, Naomi, and Ouchi, Tadashi. Relativistic two-body problem in quantum theory. Progress Theoret. Physics 6, 748–761 (1951).

The formulation of a wave-equation to describe the motion of two interacting particles in relativistic quantum theory is an old problem. Dirac [Proc. Roy. Soc. London. Ser. A. 136, 453–464 (1932)] gave a formulation, the many-time theory, in which the particle interaction is not given explicitly but is implied in a quantized electromagnetic field. The problem is to eliminate the field variables and obtain a description entirely in terms of particle coordinates. Kemmer [Helvetica Phys. Acta 10, 47–67 (1937)] proposed a wave-equation which was not derived from an exact theory but simply postulated; it turned out for various reasons to be unsatisfactory. The correct equation was finally discovered independently by Bethe and Salpeter [Physical Rev. (2) 82, 309–310 (1951)] and by Nambu [Progress Theoret. Physics 5, 82–94 (1950); these Rev. 11, 766]. The authors here have attempted to derive from the Dirac theory a wave-equation of the Kemmer type. The discussion is rather incomplete and it is not clear to the reviewer precisely what approximations have been made. The authors do not seem to be aware of the work of Nambu, Bethe and Salpeter. *F. J. Dyson* (Ithaca, N. Y.).

Thomas, L. H. The relativistic dynamics of a system of particles interacting at a distance. Physical Rev. (2) 85, 868–872 (1952).

A particular quantum dynamics of many particles is constructed which includes their mutual interaction in a relativistically invariant manner. Invariance under homogeneous Lorentz transformations is made trivial by supposing that all six angular momentum components of individual particles are additive, but for interacting particles the momenta are not additive although of course they satisfy the correct commutation relations. The world line of a particle is then in general not invariant. *H. C. Corben.*

Dirac, P. A. M. A new classical theory of electrons. Proc. Roy. Soc. London. Ser. A. 209, 291–296 (1951).

The author considers that current renormalization methods in quantum electrodynamics result in a theory which is ugly and incomplete. He believes that the primary source of our difficulties should be ascribed "not to a fault in the general principles of quantization, but to our working with a wrong classical theory". He therefore proposes a new theory for the motion of electrons which is simpler than that of Lorentz and provides exact equations of motion without making any assumptions about the structure of the electron. If A_μ is the electromagnetic four-potential and (I) $F_{\mu\nu}=A_{\mu,\nu}-A_{\nu,\mu}$, where the comma denotes differentiation, then instead of imposing the Lorentz condition $A^\mu{}_{,\mu}=0$ to eliminate gauge redundancy, the author assumes that (II) $A_\mu A^\mu=k^2$ where k is a universal constant. From the lagrangian $\mathfrak{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}\lambda(A_\mu A^\mu-k^2)$ he deduces the field equation (III) $F^{\mu\nu}{}_{,\nu}=\lambda A^\mu$. The five equations (II) and (III), together with the definition (I) are the fundamental equations to be satisfied by the five field quantities A_μ, λ. The current vector j_μ is identified with $-\lambda A_\mu$ and then the assumption $k=m/e$ implies that infinitesimal charges in a given field will move in a way which agrees with Lorentz's equations for charges moving in the same field neglecting the self-field of the charges. The author lays great stress on the fact that e and m enter the theory only in the ratio e/m which he would expect in a purely classical theory; for the existence of the elementary charge e should appear as a distinct quantum effect. This remark proves to be the key to the author's subsequent revision of his theory.

A. J. Coleman (Toronto, Ont.).

Schrödinger, Erwin. Dirac's new electrodynamics. Nature 169, 538 (1952).

W. Gordon [Z. Physik 40, 117–133 (1926)], O. Klein [ibid. 37, 895–906 (1926)] and E. Schrödinger [Ann. Physik (4) 81, 109–139 (1926); 82, 265–272 (1927)] independently proposed field equations for interacting vector (A_μ) and charged scalar (ψ) fields. The author shows that these equations can be simplified by a change of gauge and that with the identification $\lambda=\psi^*\psi$ four of the equations are identical with equations (III) of Dirac [see preceding review]. Dirac's equation II appears in a first approximation to the Gordon-Klein-Schrödinger theory when the spatial variation of λ is negligible in the length k^{-1}.

A. J. Coleman (Toronto, Ont.).

Kompaneec, A. On a new formulation of Dirac's electrodynamics. Doklady Akad. Nauk SSSR (N.S.) 82, 873–875 (1952). (Russian)

The field equations of Dirac's new classical electrodynamics [see the second preceding review] are

$$(\partial^k F_i/\partial x^k)=4\pi\lambda A_i, \tag{1}$$

$$A_i A^i=-k^2. \tag{2}$$

The author finds the exact solution of these equations in which the electric charge is initially at rest and distributed uniformly throughout a sphere of radius R with density ρ_0. Let

$$\phi=-k\cosh\psi, \quad E=k\epsilon \tag{3}$$

be respectively the electrostatic potential and the (radial) electric field strength, at the position r and at the time t. The solution is implicit in the following two equations relating ψ, ϵ to r, t.

$$\epsilon r+\cosh\psi=1+[4\pi\rho_0\epsilon^2 r^4/3k]^{1/3}, \tag{4}$$

$$\pm ct=\epsilon r^2\int_0^\psi \cosh\psi'\,d\psi'[\epsilon r+\cosh\psi-\cosh\psi']^{-2}. \tag{5}$$

The ambiguous sign in (5) is not fixed by (1) and (2). Physically, the negative sign must be excluded as it describes a charge-distribution collapsing to a point. The solution with positive sign describes a dispersing charge, and in the limit of small ρ_0 this solution goes over into the solution of the corresponding problem in ordinary electrodynamics.

F. J. Dyson (Ithaca, N. Y.).

Valatin, Jean G. Sur la quantification de la nouvelle théorie classique de Dirac. I. C. R. Acad. Sci. Paris 234, 64–67 (1952).

Valatin, Jean G. Sur la quantification de la nouvelle théorie classique de Dirac. II. C. R. Acad. Sci. Paris 234, 188–190 (1952).

Part I proposes a theory in which the matter field but not the electromagnetic field is quantized and which, for $h\to 0$, reduces to Dirac's new classical theory of the electron [see the third preceding review].

In part II the electromagnetic field is also quantized. However, in order to avoid certain singular surfaces, this is done by means of the Stueckelberg-de Broglie method of ascribing a non-zero mass μ to the photon, effecting the

quantization, and then taking the limit $\mu \to 0$. The author is not altogether satisfied with this since he fears that $\mu = 0$ is an essential singularity of such a nature as to invalidate this procedure. *A. J. Coleman* (Toronto, Ont.).

Friedrichs, K. O. Mathematical aspects of the quantum theory of fields. III. Comm. Pure Appl. Math. **5**, 1–56 (1952).

Using the mathematical apparatus set up in parts I–II of this series [Comm. Pure Appl. Math. **4**, 161–224 (1951); these Rev. **13**, 520], the theory of a scalar quantized field interacting with an external un-quantized source function is developed in detail. Mathematically speaking, this paper contains no ideas not already explained in parts I–II. Physically speaking, the results are the same as those given in a brief paper by R. J. Glauber [Physical Rev. **84**, 395–400 (1951)]. *F. J. Dyson* (Ithaca, N. Y.).

Neuman, Maurice. Eigenvalue problem in quantum electrodynamics. Physical Rev. (2) **85**, 129–133 (1952).

The paper contains an attempt to derive an equation for two electrons, in interaction, from a field theory in which electrons are acted upon only by external fields. The procedure is as follows: the inhomogeneous integral equation which describes the scattering of an electron in an external potential and which contains two parameters (charge and mass of the electron) is solved in terms of the Fredholm theory. The restrictions on the parameters that are necessary for the existence of a solution of the homogeneous integral equation are expressed by the fact that the Fredholm determinant must vanish. The first Fredholm minor has then all the properties of the wavefunction of the system and is identified with it. If however the first minor should vanish identically, then the integral equation is solved by the second minor (if it does not vanish), which is antisymmetric in its two indices and is therefore identified with the wavefunction for two noninteracting electrons acted upon by the external field. It is now postulated that these identifications of the first and second minor hold even when the external field is replaced by photon operations which allow for action of a particle on itself or for interactions between particles. For the actual calculation of the minors an interaction procedure is used which corresponds to an expansion in interaction patterns rather than in powers of the coupling constant. The self-energy of the electron is calculated in first pattern approximation, is found to be infinite as usual, and the ambiguity of the usual renormalization procedure to extract finite terms from this infinite result is pointed out. *T. E. Hull* (Vancouver, B. C.).

de Broglie, Louis. Sur les relations entre les coefficients de charge et de masse dans la théorie du champ soustractif. C. R. Acad. Sci. Paris **234**, 1505–1507 (1952).

In the author's subtractive field theory [same C. R. **232**, 1269–1272 (1951); these Rev. **12**, 890] an electron interacts with n neutral fields, where n is an integer greater than 1. The ith field has quanta with the rest-mass m_i, and a coupling with the electron proportional to the coupling-constant or charge e_i. In order that the theory be free of divergences it is necessary that the conditions $\sum_{i=1}^{n} e_i m_i^{2p} = 0$, $p = 0, 1, \cdots, n-2$, be satisfied. *F. J. Dyson.*

Blohincev, D. I. On the generalized reciprocity law. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **22**, 254 (1952). (Russian)

This letter is in answer to criticisms of the author's paper [same Žurnal **18**, 566–574 (1948); these Rev. **10**, 345] and earlier papers, concerned with his theory of particles with extended charge. The critics asserted that the theory would imply the possibility of transmitting signals faster than light over macroscopic space-time intervals. The author shows by a simple calculation that this is not the case. All signals either (a) travel with a velocity exceeding the light-velocity by a macroscopically undetectable amount, or (b) do not exceed the light-velocity at all, or (c) decay exponentially within a microscopic region of space-time. *F. J. Dyson.*

Matthews, P. T., and Salam, Abdus. The renormalization of meson theories. Rev. Modern Physics **23**, 311–314 (1951).

The authors of this paper have solved the difficult problem of extending the method of mass- and charge-renormalization, designed originally for quantum electrodynamics, so that it can be used in order to obtain finite and unambiguous results from certain types of meson theory. The work has been published in a series of technical papers [P. T. Matthews, Physical Rev. (2) **80**, 292–293 (1950); these Rev. **12**, 784; A. Salam, ibid. **84**, 426–431 (1951); these Rev. **13**, 608; and references there quoted]. The present paper is a clear and non-technical summary of the results of all this work, with a description of the nature of the difficulties and the methods used to overcome them. *F. J. Dyson.*

Thermodynamics, Statistical Mechanics

***Hinčin, A. Ya. Matematičeskie osnovaniya kvantovoĭ statistiki. [Mathematical foundations of quantum statistics.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951. 256 pp. 10.20 rubles.

This remarkable book is the quantum theory counterpart of the author's Mathematical foundations of statistical mechanics [Moscow-Leningrad, 1943; these Rev. **8**, 187; or in translation, Dover, New York, 1949; these Rev. **10**, 666]. In the earlier book he systematized classical statistical mechanics, reducing the calculations to the central limit theorem (with remainder) in the theory of probability. In the present book he carries through the corresponding work in quantum statistics, reducing the calculations to the local limit law of the theory of probability, that is, to the evaluation of the probabilities of the values assumed by the sum of a large number of mutually independent integer-valued random variables with a common distribution.

Chapter I contains introductory remarks on probability, and proofs of the necessary limit theorems. The results are stated in such a form that calculations in the application to quantum statistics become trivial. Chapters II and III contain a general introduction to quantum theory and quantum statistics. Chapter IV treats the statistics of systems of photons, the easiest case. Chapter V treats systems of particles of any sort (of only one kind) containing a definite number of particles. Chapter VI ties in the statistical theory with thermodynamics. Appendices treat systems containing particles of more than one kind.

The greatest stress is on systems made up of a large number N of particles, and on additive quantities connected with these systems. An additive quantity is one which is the sum of N quantities, with the ith quantity a function (not depending on i) of the energy of the ith particle. The microcanonical average and dispersion of any additive quantity is calculated, and the latter is shown to be $o(N^2)$. The latter

fact is shown to imply that in a reasonable sense the relative error of an additive quantity, measured from its microcanonical average, can be expected to be small. The elegance of the author's treatment of the subject lies in the simplicity with which the averages like those involved in such a discussion can be calculated. See also his papers [Trudy Mat. Inst. Steklov. 33 (1950); these Rev. 13, 416, and the paper reviewed below] in which he treats the distribution theory of additive quantities and a general class of non-additive quantities respectively. *J. L. Doob* (Urbana, Ill.).

***Hinčin, A. Ya. On some general theorems of statistical physics.** Trudy Mat. Inst. Steklov., v. 38, pp. 345–365. Izdat. Akad. Nauk SSSR, Moscow, 1951. (Russian) 20 rubles.

Following Jeans, the author calls a quantity in statistical mechanics "normal" if it is nearly the same for most states of the microcanonical system under consideration. The value assigned to such a quantity is then largely independent of the weights assigned to the states. For example [see the preceding review] additive quantities are normal. The author now proves that quantities of a more general class are normal. A quantity in this class is a symmetric function of the energies of the component particles of a system, where the function satisfies a rather weak condition of Lipschitz type. Using this result it is shown that if microcanonical state weights are altered in a specified, rather general, way, the limiting distribution, as the number of particles becomes infinite, of the energy distribution of the particles is unchanged. *J. L. Doob* (Urbana, Ill.).

Scheidegger, A. E., and McKay, C. D. On the thermodynamics of wave fields. Canadian J. Physics 30, 117–118 (1952).

Nanda, V. S. Partition theory and thermodynamics of multi-dimensional oscillator assemblies. Proc. Cambridge Philos. Soc. 47, 591–601 (1951).

This is a study of partition theory of numbers corresponding to the thermodynamic assemblies of two- and three-dimensional oscillators (quantized according to Bose-Einstein). The "Zustandssumme" of an assembly of a variable number of two-dimensional oscillators is shown to be identical with the generating function of plane partitions, and the two-dimensional oscillator assembly is connected with the partition of bi-partite numbers. These facts are used in obtaining asymptotic forms for the generating functions, these being later generalized to the three-dimensional case. *B. O. Koopman* (New York, N. Y.).

Green, Melville S. Brownian motion in a gas of noninteracting molecules. J. Chem. Phys. 19, 1036–1046 (1951).

The Brownian motion of a rigid spherical particle under the random impact of the non-mutually interacting point molecules of a gas is investigated by the conventional Rayleigh-Smoluchowski type of method. This leads to an integro-differential equation for the evolution in time of the probability density of the particle in its phase-space. Approximations are then used to reduce this equation to Chandrasekhar's generalized Fokker-Planck equation. With other approximating assumptions, the distribution of changes of momentum of the Brownian particle is studied in detail, an asymptotic formula being obtained. The friction constant derived from the result is found to be in agreement with that obtained by J. G. Kirkwood. It agrees with that found experimentally by Millikan for the oil droplet in its dependence on temperature, but with a somewhat lower value, apparently due to the physical difference between a droplet and an idealized rigid sphere. The standard of mathematical rigor is rather higher than in conventional treatments of the subject. There are many misprints; and an additional difficulty is produced by the author's habit of omitting many parentheses from formulas, thereby replacing correct ones by incorrect ones. *B. O. Koopman.*

MacLellan, A. G. A statistical-mechanical theory of surface tension. Proc. Roy. Soc. London. Ser. A. 213, 274–284 (1952).

***Kampé de Fériet, J. Statistical mechanics of a continuous medium (vibrating string with fixed ends).** Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 553–566. University of California Press, Berkeley and Los Angeles, 1951. $11.00.

In order to ultimately treat turbulence by statistical mechanics, a simpler system, also of infinite degrees of freedom, is considered. The problem is to establish a phase space, a flow of the space into itself, an ergodic theorem (existence of time averages along a trajectory), an invariant measure, and to determine the indecomposable (metrically transitive) subspaces.

Consider $y_{xx}=y_{tt}$, $0\leqq x\leqq l$, $y(0,t)=y(l,t)=0$, $v=y_t$. At a given instant, the state of the string, $y(x)$, $v(x)$, can be represented by a single $2l$-periodic function $f(x)$: $f(x)=\frac{1}{2}[y'(x)+v(x)]$, $0\leqq x\leqq l$; $f(x)=\frac{1}{2}[y'(-x)-v(-x)]$, $-l\leqq x\leqq 0$; $y(x)=\int_{-x}^{x}f(s)ds$, $v(x)=f(x)-f(-x)$. The phase space is all continuously differentiable $2l$-periodic functions $f(x)$ subject to $\int_0^{2l}f(s)ds=0$. Since the solution of the differential equation is $y(x,t)=\int_{-x}^{x}f(s+t)ds$, the transformation of the phase space is $T_tf(s)=f(s+t)$. The periodicity of f implies that each trajectory is a closed curve. It follows trivially that the infinite time average exists for each trajectory; all invariant measures are evident; the only indecomposible subspaces are the individual trajectories, and time averages are equal to phase averages only in the trivial case of a measure concentrated on a single trajectory. For the proof, introduce the measure space $B\times C$, B arbitrary and C a circle, $\beta\varepsilon B$, $\gamma\varepsilon C$. To every function $g(\beta,\gamma)$ measurable in β, continuously differentiable and periodic in γ with $\int_0^{2l}g(\beta,\gamma)d\gamma=0$, a measure is defined on $f(s)=g(\beta,\gamma+s)$. A correlation function and the average values of several moments are computed. *H. Grad* (New York, N. Y.).

Adams, E. N., II. Motion of an electron in a perturbed periodic potential. Physical Rev. (2) 85, 41–50 (1952).

Nach dem Theorem von G. H. Wannier [Physical Rev. (2) 52, 191–197 (1937)] kann in einer gewissen Näherung die Bewegung eines Elektrons in einem durch ein gestörtes periodisches Potential beschriebenem Felde mit Hilfe der Schrödingergleichung

$$\left[E^0(p)+V'(r)-i\hbar\frac{\partial}{\partial t}\right]\Psi(r)=0$$

berechnet werden, in der E^0 die Energie bei ungestörtem Potential und $V'(r)$ die Störungsfunktion bedeutet. $|\Psi(r)|^2$ ist die Wahrscheinlichkeit, dass man das Elektron bei r in einer Einheitszelle findet. Vor kurzer Zeit hat J. C. Slater [ibid. 76, 1592–1601 (1949)] einige Anwendungen dieses Theorems unter der Annahme besprochen, dass erstens die Störung genügend klein dazu ist, dass man $V'(r)$ nach den

zu einem Energieband gehörenden Eigenfunktionen entwickeln kann und dass zweitens $V'(r)$ eine mit der Lage nur langsam veränderliche Funktion ist. Da jedoch bei den Metallen sich meistens zwei Energiebänder überdecken, so ist es nicht sicher, dass diese Annahmen tatsächlich erfüllt sind. In der vorliegenden Arbeit wird deshalb die Slatersche Herleitung in dem Sinne erweitert, dass auch Übergänge zwischen verschiedene Bänder berücksichtigt werden. Zu diesem Zwecke wird $\Psi(r)$ als eine Kolonnenmatrix definiert, die in jedem Band eine Komponente besitzt. Die auftretenden Korrektionsglieder werden dabei in solch eine Form gebracht, dass sie als Störungen behandelt werden können. Dadurch erhält man eine verallgemeinerte Form des Wannierschen Theorems in der die Störung so von den Koordinaten, wie von dem Impuls abhängen kann. Die Fälle des elektrischen und des magnetischen Störfeldes werden besprochen und bei dem letzteren das Resultat von J. M. Luttinger [ibid. **84**, 814–817 (1951); diese Rev. **13**, 517] verallgemeinert. Die Wechselwirkung von Elektronen und Schallquanten (Phononen) wird ebenfalls diskutiert.

T. Neugebauer (Budapest).

Brooks, J. E., and Domb, C. Order-disorder statistics. III. The antiferromagnetic and order-disorder transitions. Proc. Roy. Soc. London. Ser. A. **207**, 343–358 (1951).

This is a generalization of the previous work of one of the authors [C. Domb, same Proc. **196**, 36–50; **199**, 199–221 (1949); these Rev. **10**, 666; **12**, 68], on the high and low temperature series for thermodynamic and magnetic properties of the Ising model of a ferromagnet to the corresponding two-dimensional model of antiferromagnetic materials and binary substitutional alloys. It is suggested that no order-disorder transition occurs in a binary alloy when one of the components has a concentration of less than 0.226. The results are applied to the theory of absorption of gases on solids. *E. W. Montroll* (College Park, Md.).

Domb, C., and Potts, R. B. Order-disorder statistics. IV. A two-dimensional model with first and second interactions. Proc. Roy. Soc. London. Ser. A. **210**, 125–141 (1951).

Statistical mechanical problems in cooperative phenomena can be formulated in terms of the largest characteristic root Λ of a "duo-diagonal" matrix. If $x=\exp(-2J_s/kT)$ and $w=\exp(-2J_w/kT)$ are associated with first and second neighbor interactions and μ corresponds to the external field, the initial terms of a low temperature series for Λ in powers of μ, x and w are given for square and triangular lattices. It is stated that this series is valid for μ, x and $y<1$ and that this series has been derived by a method given in Potts' thesis [Oxford, 1951]. The case in which $J_s>0$, $J_w>0$ is the "ferromagnetic" case. For the first and second neighbor case, Potts has also given reciprocal relations which permit one to express $\Lambda(x, w)$ in terms of $\Lambda(x^{-1}, w)$ and consequently cases in which $J_s<0$ can also be considered. For $w>1$ other series are given but in somewhat fewer terms. The reciprocal relations permit an approximate treatment of transition points and approximate treatments of the partials of Λ relative to μ, x and w are also given. The effect of second neighbors in the case of ferromagnets, binary alloys, anti-ferromagnets and monolayers is discussed.

F. J. Murray (New York, N. Y.).

Potts, R. B. Some generalized order-disorder transformations. Proc. Cambridge Philos. Soc. **48**, 106–109 (1952).

The author generalizes the work of Kramers and Wannier [Physical Review (2) **60**, 252–262, 263–276 (1941); these Rev. **3**, 64] which showed the existence of a transformation S on z for the no-field partition function $f(z)$ for a square lattice such that $f(Sz)$ can be expressed in terms of $f(z)$ and z ($z=\exp(-2J/kT)$). The fixed point of S corresponds to the Curie point. The Kramers-Wannier work refers to the situation in which a unit vector, parallel or anti-parallel to a given direction, is located at each point of the lattice and the energy depends on the inner product of neighboring vectors. Here the case of two, three and four directions for the vectors is treated. *F. J. Murray* (New York, N. Y.).

Yamamoto, Tunenobu. On the crystal statistics of two-dimensional Ising ferromagnets. Progress Theoret. Physics **6**, 533–542 (1951).

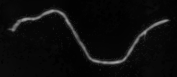
By interpreting each atom on a triangular Ising as a pair with an imaginary interaction constant the author shows that the calculation of the partition function for the two-dimensional triangular and hexagonal lattices can be reduced to that of a square lattice. The connection between the duality theorems for the above mentioned three types of lattices is given. *E. W. Montroll* (College Park, Md.).

Katsura, Shigetoshi, and Fujita, Hisaaki. Point of condensation and the volume dependency of the cluster integrals. Progress Theoret. Physics **6**, 498–505 (1951).

The authors criticize the work of Mayer and others because certain double limit considerations are not made. Thus, if $b_l(V)$ denotes the "cluster integral" for l molecules, it has been customary to use the series

$$\chi(z)=\sum_l(\lim_{V\to\infty}b_l(V))z^l \quad\text{and}\quad \zeta(z)=\sum_l l(\lim_{V\to\infty}b_l(V))z^l$$

instead of

$$\chi_\infty(z)=\lim_{V\to\infty}\sum_l b_l(V)z^l \quad\text{and}\quad \zeta_\infty(z)=\lim_{V\to\infty}\sum_l lb_l(V)z\,.$$

(z is the "fugacity"; cf. Mayer and Mayer, Statistical mechanics, Wiley, New York, 1940, p. 292.) The authors claim that they can show that $\chi(z)=\chi_\infty(z)$ for some range of z, $0\leqq z\leqq z^*$ for a z^* which does not exceed z_c, the point previously given as the point of condensation, and that χ_∞ has a singularity at z^*. They give an example of a series with this behavior which cannot be considered under the previous theory but the physical significance of the example is not clear to the reviewer. A similar counter example is given against the method of "generating functions" for $\zeta_\infty(z)$. The authors claim that the equivalent of z_c in this case is associated with the limits of supersaturation and z^* corresponds to condensation. *F. J. Murray* (New York, N. Y.).